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On the stratified classical configuration space of lattice QCD

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Abstract

The stratified structure of the configuration space $\mathbf{G}^N = G \times \cdots \times G$ reduced with respect to the action of G by inner automorphisms is investigated for $G = \mathrm{SU}(3)$. This is a finite dimensional model coming from lattice QCD. First, the stratification is characterized algebraically, for arbitrary N. Next, the full algebra of invariants is discussed for the cases $N = 1$ and $N = 2$. Finally, for $N = 1$ and $N = 2$, the stratified structure is investigated in some detail, both in terms of invariants and relations and in more geometric terms. Moreover, the strata are characterized explicitly using local cross sections.

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1. Introduction

If one wants to analyze the non-perturbative structure of gauge theories, one should start with clarifying basic structures like that of the field algebra, the observable algebra

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and the superselection structure of the Hilbert space of physical states. It is clear that the standard Doplicher–Haag–Roberts theory [1,2] for models, which do not contain massless particles, does not apply here. Nonetheless, there are interesting partial results within the framework of general quantum field theory both for quantum electrodynamics (QED) and for non-abelian models, see [3–6].

To be rigorous, one can put the system on a finite lattice, leaving the (extremely complicated task) of constructing the full continuum limit, for the time being, aside. This way, one gets rid of complicated functional analytical problems, but the gauge theoretical problems one is interested in are still present within this setting. For basic notions concerning lattice gauge theories (including fermions) we refer to [7] and references therein. Our approach is Hamiltonian, thus, we put the model on a finite (regular) cubic lattice. In this context, we have formulated (and in the meantime partially solved [8–12]) the following programme:

1. Describe the field algebra $\mathfrak{A}_\Lambda$ in terms of generators and defining relations and endow it with an appropriate functional analytical structure.
2. Describe the observable algebra $\mathfrak{O}_\Lambda$ (algebra of gauge invariant operators, fulfilling the Gauss law) in terms of generators and relations.
3. Analyze the mathematical structure of $\mathfrak{O}_\Lambda$ and endow it with an appropriate functional analytical structure.
4. Classify all irreducible representations of $\mathfrak{O}_\Lambda$.
5. Investigate dynamics in terms of observables.

Finally, of course, one wants to construct the continuum limit. As already mentioned, in full generality, this is an extremely complicated problem of constructive field theory. However, the results obtained until now suggest that there is some hope to control the thermodynamical limit, see [8] for a heuristic discussion. We also mention that for simple toy models, these problems can be solved, see [14].

In [12,13] we have started to investigate the structure of the field and the observable algebra of lattice QCD. In these papers we took the attitude of implementing the constraints on the quantum level. It is well known that there is another possibility: first, one reduces the classical phase space and then one formulates the quantum theory over this reduced phase space. Since the action of the gauge group can have several orbit types, the first step has to be done using singular Marsden–Weinstein reduction [19]. Then the reduced phase space has the structure of a stratified symplectic space. Quantization procedures for such spaces have been worked out recently or are still under investigation [20]. As an important ingredient for both reduction and quantization, in this paper, we study the stratified structure of the reduced classical configuration space. For QCD on a finite lattice, this is given by the orbit space of the action of SU(3) on $\mathrm{SU}(3)^N = \mathrm{SU}(3) \times \cdots \times \mathrm{SU}(3)$ by inner automorphisms.

Our paper is organized as follows: in Section 2 we give a precise formulation of the problem and we discuss the basic tools used in this paper. In Section 3, the stratification of the reduced configuration space is characterized algebraically for arbitrary N. Next, in Section 4 the full algebra of invariants is discussed for the cases $N = 1$ and $N = 2$. Finally, in Sections 5 and 6 the stratified structure is investigated for $N = 1$ and $N = 2$ in some detail, both in terms of invariants and relations and in more geometric terms. Moreover, the strata are characterized explicitly using local cross sections.

2. Basics

We consider QCD on a finite regular cubic lattice Λ in the Hamiltonian framework. In this context, the classical gluonic potential is approximated by its parallel transporter:

$$\Lambda^1 \ni (x, y) \to g_{(x,y)} \in G,$$

where $G = \mathrm{SU}(3)$ and Λ^1 denotes the set of one-dimensional elements (links) of Λ. Thus, the classical configuration space $\mathcal{C}_{(x,y)}$ over a given link (x, y) is isomorphic to the group manifold G and the classical phase space over (x, y) is isomorphic to

$$T^*G \cong \mathfrak{g}^* \times G.$$

Thus, the (gluonic) lattice configuration space is given by

$$\mathcal{C}_\Lambda = \prod_{(x,y)\in\Lambda^1} \mathcal{C}_{(x,y)}. \tag{2.1}$$

It is obviously isomorphic to the product

$$\mathbf{G}^L := \underbrace{G \times \cdots \times G}_{L},$$

with L denoting the number of lattice links. The corresponding phase space is a product of phase spaces of the above type. Gauge transformations act on parallel transporters by

$$g_{(x,y)} \mapsto g'_{(x,y)} = g_x \cdot g_{(x,y)} \cdot g_y^{-1},$$

with

$$\Lambda^0 \ni x \mapsto g_x \in G$$

and Λ^0 denoting the set of zero-dimensional elements (sites) of Λ. These transformations induce transformations of the phase space over (x, y). Thus, the lattice gauge group is given by

$$G_\Lambda = \prod_{x\in\Lambda^0} G_x, \tag{2.2}$$

with every G_x being a copy of G.

The above symmetry can be easily reduced using the following technique: we choose a *lattice tree*, which consists of a fixed lattice point (root) x_0 and a subset of Λ^1 such that for every lattice site x there is a unique lattice path from x to x_0. Now, we can fix the gauge on every on-tree link and we can parallel transport every off-tree configuration variable to the point x_0. This can be viewed as a reduction with respect to the pointed lattice gauge group

$$G_{\Lambda}^{0} = \prod_{x_0 \neq x \in \Lambda^0} G_x. \tag{2.3}$$

We end up with a partially reduced configuration space being isomorphic to $\mathbf{G}^N$, with N denoting the number of off-tree links. The corresponding phase space is given by the cotangent bundle $T^*\mathbf{G}^N$. The reduced gauge group is $G_{x_0} \equiv G$, acting via inner automorphisms $G \ni g \mapsto \mathrm{Ad}_g \in \mathrm{Aut}(\mathbf{G}^N)$:

$$\mathrm{Ad}_g(g_1, \ldots, g_N) = (g \cdot g_1 \cdot g^{-1}, g \cdot g_2 \cdot g^{-1}, \ldots, g \cdot g_N \cdot g^{-1}).$$

Thus, we have a finite dimensional Hamiltonian system with symmetry group G. Since this action has several orbit types, quantization turns out to be a complicated task. Usually, the non-generic strata occurring here are omitted. If one wants to include them consistently, one has to develop a quantum theory over a stratified set. One option to do this is to perform quantization after reduction, i.e., to quantize the reduced phase space of $\mathbf{G}^N$. This is a stratified symplectic space which is constructed from $T^*\mathbf{G}^N$ by singular Marsden–Weinstein reduction [19]. By properly implementing the tree gauge on the level of the phase space, it can be shown that this space is isomorphic, as a stratified symplectic space, to the reduced phase space of the full lattice gauge theory [18]. This completely justifies the use of the tree gauge in this approach. The reduced phase space of $\mathbf{G}^N$ is a bundle over the reduced configuration space

$$\hat{\mathcal{C}}_{\Lambda} \cong \mathbf{G}^N/\mathrm{Ad}_G. \tag{2.4}$$

In this work, we investigate $\hat{\mathcal{C}}_{\Lambda}$ for $N = 1$ and 2.

Our strategy is as follows:

(i) It is well known that orbit types of the action of a Lie group G on a manifold M are classified by conjugacy classes of stabilizers $[G_m]$, $m \in M$, of the group action. Moreover, the orbit of an element m is diffeomorphic to G/G_m. Thus, in Section 3, we list the orbit types by calculating their stabilizers. This is done for *arbitrary* N. Moreover, all orbit types will be characterized algebraically, in terms of properties of eigenvectors and eigenvalues of representatives.

(ii) Next, in order to investigate the geometric structure of $\hat{\mathcal{C}}_{\Lambda}$, we make use of basic facts from invariant theory. According to [16], if we have an action of a Lie group G on a manifold M with a finite number of orbit types, then the orbit space of this action can be characterized as follows: let $(\rho_1 \cdots \rho_p)$ be a set of generators of the algebra of invariant polynomials of the G-action on M. They define a mapping

$$\rho = (\rho_1 \cdots \rho_p) : M \longrightarrow \mathbb{R}^p,$$

which induces a homeomorphism of the orbit space $X := M/G$ onto the image of ρ in $\mathbb{R}^p$. Next, restricting our attention to the case of G being an $(n \times n)$-matrix group and $M = \mathbf{G}^N$, we can use general results as developed in [15]: the algebra of polynomials, which are invariant under simultaneous conjugation of N matrices is generated by traces of products of these matrices,

$$\mathbf{G}^N \ni (g_1, \dots, g_N) \mapsto \mathrm{tr}(g_{i_1} g_{i_2} \cdots g_{i_k}) \in \mathbb{C}, \tag{2.5}$$

with $k \leq 2^n - 1$. Moreover, for $\mathrm{Gl}(n, \mathbb{R})$, the full set of relations between generators is given by the so-called fundamental trace identity

$$\sum_{\sigma \in S_{n+1}} \mathrm{sgn}(\sigma) \cdot \prod_{(i_1,\dots,i_j)} \mathrm{tr}(g_{i_1} \cdots g_{i_j}) = 0, \tag{2.6}$$

where $(i_1, \dots, i_j)$ ranges over the set of all cycles of the cycle decomposition of the permutation σ. In the case under consideration, $G = \mathrm{SU}(n)$, we have two additional relations induced from the two invariant tensors of $\mathrm{SU}(n)$, see [21],

$$\mathrm{tr}(gg^\dagger) = n, \tag{2.7}$$

$$\det(g) = 1. \tag{2.8}$$

Relations (2.7) and (2.8) imply the following form of the characteristic polynomial of $g \in G = \mathrm{SU}(3)$:

$$\chi_g(\lambda) = \lambda^3 - \mathrm{tr}(g)\lambda^2 + \overline{\mathrm{tr}(g)}\lambda - 1. \tag{2.9}$$

The above listed facts enable us to characterize the configuration space in terms of invariant generators and relations. First, in Section 4, we investigate the algebra of invariants and their relations. Next, in Sections 5 and 6.1 we study the mapping ρ in some detail. For $N = 1$ we solve the problem completely, that means we find the range of ρ and characterize $\hat{\mathcal{C}}_\Lambda$ as a compact subset of $\mathbb{R}^2$. For $N = 2$, we will find a unique characterization of each orbit type in terms of invariants. But to find the range of ρ, defined in terms of a number of *inequalities* between invariants, turns out to be a complicated problem. Therefore, this will be discussed in a separate paper, see [22]. There, we will present a complete topological characterization of $\hat{\mathcal{C}}_\Lambda$ for $N = 2$ as a CW-complex.

(iii) We present a somewhat detailed geometric characterization of all occurring strata in terms of subsets and quotients of SU(3), see Section 6.2.

(iv) Using a principal bundle atlas of SU(3), viewed as an SU(2)-bundle over S^5, we construct representatives of orbits for all occurring strata, see Section 6.3.

3. The stratification of the configuration space

First, let us consider the case $N = 1$.

Theorem 3.1. *The adjoint action of* SU(3) *on* $\mathbf{G}^1 \equiv \mathrm{SU}(3)$ *has three orbit types, corresponding to three conjugacy classes of stabilizers of dimensions* 2, 4 *and* 8, *respectively. The orbit space* $\mathbf{G}^1/\mathrm{Ad}_{\mathrm{SU}(3)}$ *decomposes into three strata characterized by the following conditions*:

(1) *If g has three different eigenvalues then its stabilizer is $U(1) \times U(1)$ and g belongs to the generic stratum.*
(2) *If g has two different eigenvalues then its stabilizer is $U(2)$.*
(3) *If g has only one eigenvalue then it belongs to the centre $\mathcal{Z}$ and its stabilizer is $G = \mathrm{SU}(3)$.*

Proof. Up to conjugacy, we may assume that $g = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$. In case 1, the λ_i are pairwise distinct. Hence, the stabilizer of g is

$$H_g = \{\mathrm{diag}(\alpha, \beta, \gamma) | \alpha, \beta, \gamma \in U(1), \alpha \cdot \beta \cdot \gamma = 1\} \cong U(1) \times U(1). \tag{3.1}$$

In case 2, up to conjugacy, $\lambda_1 \neq \lambda_2 = \lambda_3$. Then the stabilizer of g is

$$H_g = \left\{ \left[\begin{array}{c|c} (\det V)^{-1} & \\ \hline & V \end{array} \right] \middle| V \in U(2) \right\} \cong U(2). \tag{3.2}$$

In case 3, $\lambda_1 = \lambda_2 = \lambda_3$, i.e., g is a multiple of the identity. Hence, its stabilizer is $G = \mathrm{SU}(3)$. Finally, it is clear that cases 1–3 exhaust all possible values of the λ_i. □

Next, we deal with the general case.

Theorem 3.2. *The adjoint action of* SU(3) *on* $\mathbf{G}^N$, $N \geq 2$, *has five orbit types, corresponding to five conjugacy classes of stabilizers of dimensions* 0, 1, 2, 4 *and* 8, *respectively. The orbit space* $\mathbf{G}^N/\mathrm{Ad}_{\mathrm{SU}(3)}$ *decomposes into five strata characterized by the following conditions. Denote* $\mathbf{g} := (g_1, \ldots, g_N)$.

1. *If $g_1, \ldots, g_N$ have no common eigenspace then the stabilizer of* $\mathbf{g}$ is $H_{\mathbf{g}} = \mathcal{Z}$ *and* $\mathbf{g}$ *belongs to the generic stratum.*
2. *If $g_1, \ldots, g_N$ have exactly one common one-dimensional eigenspace then $H_{\mathbf{g}} \cong U(1)$.*
3. *If $g_1, \ldots, g_N$ have three (different) common (one-dimensional) eigenspaces then $H_{\mathbf{g}} \cong U(1) \times U(1)$.*
4. *If $g_1, \ldots, g_N$ have a common two-dimensional eigenspace then $H_{\mathbf{g}} \cong U(2)$.*
5. *If $g_1, \ldots, g_N$ have a three-dimensional common eigenspace, i.e., if they all are proportional to the identity then $H_{\mathbf{g}} = G = \mathrm{SU}(3)$.*

Proof. If there are two eigenvectors e_1 and e_2, common for all matrices $g_1, \ldots, g_N$, then also their vector product $e_1 \times e_2$ is a common eigenvector. If e_1 and e_2 are not orthogonal, then the two-dimensional space $\mathbf{P}$ spanned by them is a common eigenspace. This means that the pair (e_1, e_2) can be replaced by any orthonormal basis of $\mathbf{P}$. This implies that if $\mathbf{g}$ is not of type 1 or 2, its elements can be jointly diagonalized. We conclude that the above types exhaust all possible cases.

Next we calculate the stabilizer for each case.

1. Assume that the stabilizer of $\mathbf{g}$ contains an element $s \notin \mathcal{Z}$. Then s has at least two different eigenvalues. One of these must be non-degenerate. Since the corresponding eigenspace

is left invariant by all g_i and since it is one-dimensional, it is an eigenspace of all g_i, in contradiction to the assumption.

2. Since the g_i have a common eigenvector e_1, up to conjugacy, we may assume that

$$g_i = \left[\begin{array}{c|c} a_i & 0 \\ \hline 0 & B_i \end{array}\right],$$

where $B_i \in U(2)$. Then $H_{\mathbf{g}}$ contains the subgroup

$$\left\{ \left[\begin{array}{c|c} \alpha & 0 \\ \hline 0 & \beta\mathbf{1} \end{array}\right] \middle| \alpha, \beta \in U(1), \beta^2 = \overline{\alpha} \right\} \cong U(1). \tag{3.3}$$

Conversely, let $s \in H_{\mathbf{g}}$. Since the common eigenspace of the g_i is one-dimensional, e_1 is also an eigenvector of s. Then

$$s = \left[\begin{array}{c|c} \alpha & 0 \\ \hline 0 & A \end{array}\right],$$

where $A \in U(2)$. Again up to conjugacy, we may assume that $A = \mathrm{diag}(\beta, \gamma)$. If $\beta \neq \gamma$ then the B_i must also be diagonal, because they commute with A. Then the g_i have more than one common eigenspace, which contradicts the assumption. Hence $\beta = \gamma$ and $H_{\mathbf{g}}$ coincides with the subgroup (3.3).

3. Choose a basis in $\mathbb{C}^3$, which jointly diagonalizes all the matrices $g_1, \ldots, g_N$,

$$g_i = \begin{bmatrix} a_i & 0 & 0 \\ 0 & b_i & 0 \\ 0 & 0 & c_i \end{bmatrix}.$$

The non-existence of a two-dimensional eigenspace means that none among the three equations $a_i = b_i$, $b_i = c_i$ and $c_i = a_i$, is fulfilled for all i. This implies that any matrix which commutes with all matrices $g_1, \ldots, g_N$ must be diagonal, too. Whence, the stabilizer $H_{\mathbf{g}}$ is of the form (3.1).

4. The orthogonal complement of the two-dimensional common eigenspace of the g_i is a one-dimensional common eigenspace. Thus, up to conjugacy,

$$g_i = \left[\begin{array}{c|c} a_i & 0 \\ \hline 0 & b_i\mathbf{1} \end{array}\right]$$

and $H_{\mathbf{g}}$ contains the subgroup (3.2). Conversely, let $s \in H_{\mathbf{g}}$. The non-existence of a three-dimensional eigenspace means that there is i_0 such that $a_{i_0} \neq b_{i_0}$. Then

$$s = \left[\begin{array}{c|c} (\det V)^{-1} & 0 \\ \hline 0 & V \end{array} \right],$$

with $V \in U(2)$. Whence, $H_{\mathbf{g}}$ coincides with the subgroup (3.2).
5. In this case, all matrices $g_1, \ldots, g_N$ belong to $\mathcal{Z}$, so the statement is obvious. □

Observe that types 1 and 3 may be uniquely characterized as follows.

Corollary 3.3.

1. *The matrices* $g_1, \ldots, g_N$ *have no common eigenvector if and only if there exists a pair* (g_i, g_j) *or a triple* (g_i, g_j, g_k) *of elements not possessing any common eigenvector*.
2. *Suppose that* $g_1, \ldots, g_N$ *have three (different) common (one-dimensional) eigenspaces. There does not exist a common two-dimensional eigenspace if and only if there exists an element* g_i *with three different eigenvalues or a pair* (g_i, g_j) *such that each of its elements has exactly two different eigenvalues and non-degenerate eigenvalues correspond to different eigenvectors*.

Proof.

1. If there exists a pair (g_i, g_j) or a triple (g_i, g_j, g_k) having no common eigenvector then, obviously, there is no common eigenvector for all of them. Conversely, assume that every triple (g_i, g_j, g_k) has a common eigenvector. We prove that in this case there exists a common eigenvector for all matrices $g_1, \ldots, g_N$. First, observe that it is sufficient to consider the case when none of the matrices $g_1, \ldots, g_N$ is fully degenerate (i.e. $g_i \notin \mathcal{Z}$). This means that every g_i has at least two different eigenvalues.

   The proof goes via induction: for $K \geq 3$ we show that if any subset of $\mathbf{g}$ of K elements has a common eigenvector, then the same is true for any subset of $K+1$ elements. Thus, take a subset $(g_1, \ldots, g_{K+1})$. For each $i = 1, \ldots, K+1$, skip g_i and choose a common eigenvector v_i of the remaining set of K elements. If there exist $i \neq j$ such that v_i and v_j are parallel then they both are common eigenvectors of $g_1, \ldots, g_{K+1}$. Otherwise, there exist $i \neq j$ such that v_i and v_j are not orthogonal, because there cannot be more than 3 mutually orthogonal vectors in $\mathbb{C}^3$. Suppose that v_K and v_{K+1} is such a pair. It spans a two-dimensional subspace $\mathbf{P} \subset \mathbb{C}^3$. Since v_K, v_{K+1} are common, non-orthogonal eigenvectors of $g_1, \ldots, g_{K-1}$, $\mathbf{P}$ is a common eigenspace of these elements. Now consider v_1. Since it is an eigenvector of g_2 and since, by assumption, g_2 is not proportional to the identity, v_1 must either belong to $\mathbf{P}$ or be orthogonal to $\mathbf{P}$. But in both cases it is also an eigenvector of g_1 and, therefore, a common eigenvector of $g_1, \ldots, g_{K+1}$.
2. In this case all matrices $g_1, \ldots, g_N$ can be jointly diagonalized. If one of them has three different eigenvalues (i.e., it has no two-dimensional eigenspace), then there is no common two-dimensional eigenspace $\mathbf{P}$ for all of them. Suppose that this is not the case, i.e., that every g_i has a two-dimensional eigenspace $\mathbf{P}_i$. There will be no *common* two-dimensional eigenspace if and only if there exist i, j such that $\mathbf{P}_i \neq \mathbf{P}_j$. Then also the non-

degenerate eigenspaces $\mathbf{Q}_i$ and $\mathbf{Q}_j$ of g_i and g_j do not coincide, because they are given by the orthogonal complements of $\mathbf{P}_i$ and $\mathbf{P}_j$, respectively. Hence, the decomposition of $\mathbb{C}^3$ into common eigenspaces of g_i and g_j is $\mathbf{Q}_i \oplus \mathbf{Q}_j \oplus \mathbf{P}_i \cap \mathbf{P}_j$. □

4. The algebra of invariants

In this section, we analyze the algebra of invariants for $N = 1$ and $N = 2$. We start with invariant monomials built from one matrix.

Lemma 4.1. *The invariants* $\mathrm{tr}(g^i)$ *can be uniquely expressed in terms of* $\mathrm{tr}(g)$, *for any integer* i.

Proof. Recall formula (2.9) for the characteristic polynomial of $g \in \mathrm{SU}(3)$:

$$\chi_g(\lambda) = \lambda^3 - \mathrm{tr}(g)\lambda^2 + \overline{\mathrm{tr}(g)}\lambda - 1.$$

Thus, by the Cayley–Hamilton theorem, we have

$$g^3 - \mathrm{tr}(g)g^2 + \overline{\mathrm{tr}(g)}g - \mathbb{1} = 0. \tag{4.1}$$

Multiplying both sides of (4.1) by g^{-1} we obtain:

$$g^2 - \mathrm{tr}(g)g + \overline{\mathrm{tr}(g)} - g^{-1} = 0. \tag{4.2}$$

Taking the trace of both sides we get

$$\mathrm{tr}(g^2) = (\mathrm{tr}(g))^2 - 2\overline{\mathrm{tr}(g)}. \tag{4.3}$$

Analogously, multiplying (4.1) by g^i, $i \geq 1$ and taking the trace one gets formulae for $\mathrm{tr}(g^{i+2})$ in terms of traces of $\mathrm{tr}(g^{i+1})$, $\mathrm{tr}(g^i)$ and $\mathrm{tr}(g)$. So by induction $\mathrm{tr}(g^i)$ is uniquely given by $\mathrm{tr}(g)$. For negative i, the statement now follows from 2.7. □

So in case $N = 1$, the algebra of invariant polynomials has only two generators: the real and imaginary parts of $\mathrm{tr}(g)$. The case $N = 2$ is more complicated. Its characterization in terms of invariant generators will be given in Theorem 4.4.

Lemma 4.2. *The invariants* $\mathrm{tr}(g^i h^j)$ *can be uniquely expressed in terms of the following set of independent invariants*:

$$\{\mathrm{tr}(g), \mathrm{tr}(h), \mathrm{tr}(gh), \mathrm{tr}(g^2h)\}. \tag{4.4}$$

Proof. First, substituting $g \to gh$ in (4.2) and multiplying both sides by g^{-1} to the left we get:

$$hgh - \mathrm{tr}(gh)h + \overline{\mathrm{tr}(gh)}g^{-1} - (ghg)^{-1} = 0. \tag{4.5}$$

Taking the trace of both sides yields:

$$\mathrm{tr}(gh^2) - \mathrm{tr}(gh)\mathrm{tr}(h) + \overline{\mathrm{tr}(gh)\mathrm{tr}(g)} - \overline{\mathrm{tr}(g^2h)} = 0. \tag{4.6}$$

Thus, from five traces occurring in this equation only four are independent. In what follows, we express $\mathrm{tr}(gh^2)$ in terms of the set

$$\{\mathrm{tr}(g), \mathrm{tr}(h), \mathrm{tr}(gh), \mathrm{tr}(g^2h)\}.$$

Multiplying (4.1) by hg^i and taking the trace we obtain

$$\mathrm{tr}(hg^{i+3}) - \mathrm{tr}(g)\mathrm{tr}(hg^{i+2}) + \overline{\mathrm{tr}(g)}\mathrm{tr}(hg^{i+1}) - \mathrm{tr}(hg^i) = 0. \tag{4.7}$$

This equation enables us to express $\mathrm{tr}(hg^{i+3})$ in terms of $\mathrm{tr}(hg^{i+2})$, $\mathrm{tr}(hg^{i+1})$ and $\mathrm{tr}(hg^i)$, so by induction it can be expressed in terms of $\mathrm{tr}(hg^2)$, $\mathrm{tr}(hg)$, $\mathrm{tr}(h)$ and $\mathrm{tr}(g)$.

Starting now from an arbitrary invariant of the form $\mathrm{tr}(g^i h^j)$, we can use the above procedure recursively. First, we lower the power i of g and then we lower the power j of h. We end up with invariants of the form $\mathrm{tr}(h^m g^l)$, with $k \leq 2$, $l \leq 2$. So, to finish the proof it is sufficient to express $\mathrm{tr}(g^2h^2)$ in terms of the set (4.4). For that purpose, we use the fundamental trace identity (2.6) for $k = 4$. Substituting $g_1 = g_2 = g$, $g_3 = g_4 = h$ we obtain:

$$\begin{aligned}&\mathrm{tr}^2(g)\mathrm{tr}^2(h) - 4\mathrm{tr}(hg)\mathrm{tr}(g)\mathrm{tr}(h) - \mathrm{tr}^2(g)\mathrm{tr}(h^2) - \mathrm{tr}(g^2)\mathrm{tr}^2(h) + 2\mathrm{tr}^2(hg)\\ &\quad + 4\mathrm{tr}(g)\mathrm{tr}(h^2g) + \mathrm{tr}(h^2)\mathrm{tr}(g^2) + 4\mathrm{tr}(h)\mathrm{tr}(hg^2) - 2\mathrm{tr}(hghg) - 4\mathrm{tr}(h^2g^2) = 0.\end{aligned} \tag{4.8}$$

Using equation (4.3) we get

$$\mathrm{tr}(hghg) = \mathrm{tr}((hg)^2) = \mathrm{tr}^2(hg) - 2\overline{\mathrm{tr}(hg)}.$$

This way we obtain a formula for $\mathrm{tr}(h^2g^2)$ in terms of invariants (4.4). □

Lemma 4.3. *The invariants* $\mathrm{tr}(h^2g^2hg)$ *and* $\mathrm{tr}(h^2ghg^2)$ *have the following properties*:

1. *The sum* $\mathrm{tr}(h^2g^2hg) + \mathrm{tr}(h^2ghg^2)$ *can be expressed as a polynomial in invariants of order* $k \leq 5$,
2. $\mathrm{Re}(\mathrm{tr}(h^2g^2hg) - \mathrm{tr}(h^2ghg^2)) = 0$,
3. $\mathrm{tr}(h^2g^2hg) - \mathrm{tr}(h^2ghg^2) = \frac{1}{3}\mathrm{tr}((hg - gh)^3) = \det(hg - gh)$,
4. *The invariant* $(\mathrm{tr}(h^2g^2hg) - \mathrm{tr}(h^2ghg^2))^2$ *can be expressed as a polynomial in the invariants* (4.4) *and their complex conjugates*.

Proof.

1. Using the fundamental trace identity (2.6) for $k = 4$ and $g_1 = hgh$, $g_2 = g$, $g_3 = h$, $g_4 = g$ we obtain:

$$\begin{aligned}&2\mathrm{tr}(h^2ghg^2) + 2\mathrm{tr}(h^2g^2hg) + 2\mathrm{tr}(hghghg)\\ &\quad = \mathrm{tr}(h^2g)\mathrm{tr}(g)^2\mathrm{tr}(h) - 2\mathrm{tr}(hghg)\mathrm{tr}(g)\mathrm{tr}(h) - 2\mathrm{tr}(h^2g)\mathrm{tr}(g)\mathrm{tr}(hg)\\ &\qquad - \mathrm{tr}(h^2g)\mathrm{tr}(h)\mathrm{tr}(g^2) - \mathrm{tr}(h^3g)\mathrm{tr}(g)^2 + 2\mathrm{tr}(hghg)\mathrm{tr}(hg) + 4\mathrm{tr}(h^2ghg)\mathrm{tr}(g)\\ &\qquad + 2\mathrm{tr}(h^2g)\mathrm{tr}(hg^2) + \mathrm{tr}(h^3g)\mathrm{tr}(g^2) + 2\mathrm{tr}(hghg^2)\mathrm{tr}(h).\end{aligned} \tag{4.9}$$

On the left-hand side of this equation there are invariants of order 6, and on the right-hand side all the invariants are of lower order. By Lemma 4.1, we express $\mathrm{tr}(hghghg)$ as follows:

$$\mathrm{tr}(hghghg) = \mathrm{tr}((hg)^3) = \mathrm{tr}^3(hg) - 3\overline{\mathrm{tr}(hg)}\mathrm{tr}(hg) + 3.$$

Moving it to the right-hand side yields the statement.

2. By substituting $g \to gh$, $h \to hg$ in (4.5) we obtain:

$$\mathrm{tr}(h^2ghg^2) - \mathrm{tr}(h^2g^2)\mathrm{tr}(hg) + \overline{\mathrm{tr}(h^2g^2)\mathrm{tr}(hg)} - \overline{\mathrm{tr}(h^2g^2hg)} = 0.$$

Taking the real part yields:

$$\mathrm{Re}(\mathrm{tr}(h^2ghg^2)) - \mathrm{Re}(\mathrm{tr}(h^2g^2hg)) = 0.$$

3. The first equality is obtained by expanding the right-hand side. The second one follows from the formula for the determinant of a 3×3-matrix A in terms of traces,

$$\det(A) = \tfrac{1}{3}\mathrm{tr}(A^3) - \tfrac{1}{2}\mathrm{tr}(A^2)\mathrm{tr}(A) + \tfrac{1}{6}\mathrm{tr}(A)^3.$$

Nevertheless, it can be checked by direct computation.

4. The explicit formula expressing this invariant in terms of invariants (4.4) is lengthy and, therefore, we give it in Appendix B, including some remarks how to derive it. □

Theorem 4.4. *Any function on $\mathbf{G}^2 = G \times G$ invariant with respect to the adjoint action of G can be expressed as a polynomial in the following invariants and their complex conjugates*:

$$\begin{aligned} &T_1(g,h) := \mathrm{tr}(g), \qquad T_2(g,h) := \mathrm{tr}(h), \qquad T_3(g,h) := \mathrm{tr}(hg),\\ &T_4(g,h) := \mathrm{tr}(hg^2), \qquad T_5(g,h) := \mathrm{tr}(h^2g^2hg) - \mathrm{tr}(h^2ghg^2). \end{aligned} \tag{4.10}$$

Moreover, for given values of $T_1, \ldots, T_4$, there are at most two possible values of T_5.

Proof. First we observe that using Eq. (4.2) we can express g^{-1} in terms of positive powers of g and $\mathrm{tr}(g)$. This implies that every invariant can be expressed as a polynomial in traces of products of only positive powers of matrices g and h.

From the general theory [15] we know that we can restrict ourselves to invariants of order $k \leq 2^n - 1 = 7$. By Lemmas 4.1 and 4.2, all invariants of the type $\mathrm{tr}(g^k)$, $\mathrm{tr}(h^k)$, $\mathrm{tr}(h^ig^j)$ can be expressed in terms of T_1, T_2, T_3, T_4. Observe that all invariants of order $k \leq 3$ are of this type. In what follows we list invariants of order $k \leq 7$ which are not of this type, and for each order k we present the method of expressing it in terms of invariants of lower order and T_i.

- $k = 4$: $\mathrm{tr}(hghg)$. By Lemma 4.1, we have $\mathrm{tr}(hghg) = \mathrm{tr}((hg)^2) = \mathrm{tr}^2(hg) - 2\overline{\mathrm{tr}(hg)}$.
- $k = 5$: $\mathrm{tr}(hghg^2)$, $\mathrm{tr}(h^2ghg)$. Substituting $h \to hg$ in (4.6) we obtain:

$$\mathrm{tr}(g^2hgh) = \mathrm{tr}(g \cdot hg \cdot hg) = \mathrm{tr}(g \cdot hg)\mathrm{tr}(hg) - \overline{\mathrm{tr}(g \cdot hg)\mathrm{tr}(g)} + \overline{\mathrm{tr}(g^2 \cdot hg)}.$$

Analogously we deal with $\mathrm{tr}(h^2ghg)$.

- $k = 6$: $\mathrm{tr}(h^3ghg)$, $\mathrm{tr}(g^3hgh)$, $\mathrm{tr}(h^2g^2hg)$, $\mathrm{tr}(h^2ghg^2)$, $\mathrm{tr}(hghghg)$. The invariant $\mathrm{tr}(hghghg) = \mathrm{tr}((hg)^3)$ can be expressed in terms of $\mathrm{tr}(hg)$ by Lemma 4.1. Next, by Lemma 4.2, we can reduce the power in $\mathrm{tr}(h^3 \cdot ghg)$ and express it in terms of $\mathrm{tr}(h^2 \cdot ghg)$ and other invariants of lower order. (More precisely, we substitute $h \to ghg$ into Eq. (4.7) for $i = 0$.) We deal with $\mathrm{tr}(g^3hgh)$ analogously. Next, we rewrite $\mathrm{tr}(h^2g^2gh)$ and $\mathrm{tr}(h^2ghg^2)$ in the following way:

$$\begin{aligned}\mathrm{tr}(h^2g^2gh) &= \tfrac{1}{2}(\mathrm{tr}(h^2g^2gh) + \mathrm{tr}(h^2ghg^2)) + \tfrac{1}{2}(\mathrm{tr}(h^2g^2hg) - \mathrm{tr}(h^2ghg^2))\\ &= \tfrac{1}{2}(\mathrm{tr}(h^2g^2hg) + \mathrm{tr}(h^2ghg^2)) + \tfrac{1}{2}T_5(g,h),\\ \mathrm{tr}(h^2ghg^2) &= \tfrac{1}{2}(\mathrm{tr}(h^2g^2hg) + \mathrm{tr}(h^2ghg^2)) - \tfrac{1}{2}T_5(g,h).\end{aligned} \tag{4.11}$$

By Lemma 4.3 the sum $\mathrm{tr}(h^2g^2hg) + \mathrm{tr}(h^2ghg^2)$ can be expressed as a polynomial in invariants of lower order.

- $k = 7$: there are two types of non-trivial invariants in this case:

1. $\mathrm{tr}(h^ig^jh^lg^m)$, $i + j + l + m = 7$. If one of the powers i, j, l, m, is equal to 3 or more, we can decrease the order by an appropriate substitution in Eq. (4.7). Next, we observe that there are only two possible cases when all powers i, j, l, m are smaller than 3, namely $\mathrm{tr}(h^2g^2h^2g)$ and $\mathrm{tr}(h^2g^2hg^2)$. Substituting $h \to h^2g$ into Eq. (4.6) we obtain:

$$\begin{aligned}\mathrm{tr}(h^2g^2h^2g) &= \mathrm{tr}(g \cdot h^2g \cdot h^2g)\\ &= \mathrm{tr}(g \cdot h^2g)\mathrm{tr}(h^2g) - \overline{\mathrm{tr}(g \cdot h^2g)\mathrm{tr}(g)} + \overline{\mathrm{tr}(g^2 \cdot h^2g)}.\end{aligned}$$

   Analogously we deal with $\mathrm{tr}(h^2g^2hg^2)$.

2. $\mathrm{tr}(h^2ghghg)$, $\mathrm{tr}(g^2hghgh)$. By Lemma 4.2 we can express $\mathrm{tr}(h^2ghghg) = \mathrm{tr}(h \cdot (hg)^3)$ in terms of $\mathrm{tr}(h \cdot (hg)^2)$, $\mathrm{tr}(h \cdot (hg))$, $\mathrm{tr}(h)$ and $\mathrm{tr}(hg)$. For $\mathrm{tr}(g^2hghgh)$, we get an analogous expression.

Finally, by Lemma 4.3, $T_5(g, h)$ is purely imaginary and $(T_5(hg))^2$ can be expressed as a polynomial in T_1, T_2, T_3, T_4, so only the sign of T_5 remains undetermined. □

5. The configuration space for $N = 1$

Applying the theory outlined above is trivial for $N = 1$: from Lemma 4.1 we immediately get that the orbit space is uniquely characterized by the trace function, because it generates the algebra of invariants. Here, we will explicitly find the image of the Hilbert mapping

$$\rho : \mathrm{SU}(3)/\mathrm{Ad}_{\mathrm{SU}(3)} \to \mathbb{C} \cong \mathbb{R}^2,$$

which is simply given by the trace function, $\rho = \mathrm{tr}$.

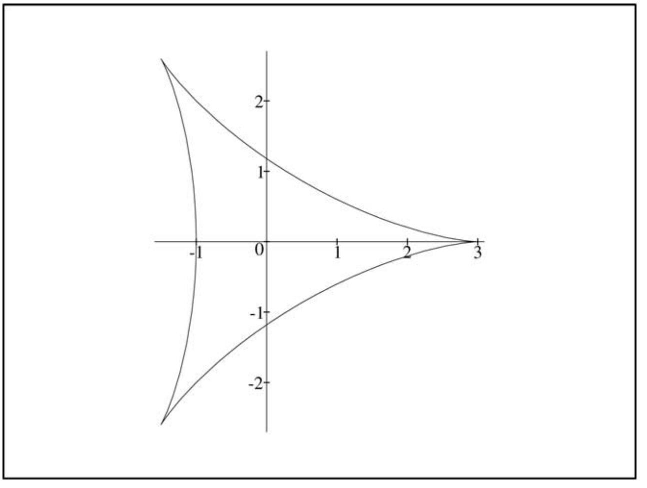

Fig. 1. Hypocycloid.

First, observe that the set of possible values of tr(g), is given by the sum of the eigenvalues of g:

$$\mathrm{tr}(g) \equiv T(\alpha, \beta) = \mathrm{e}^{\mathrm{i}\alpha} + \mathrm{e}^{\mathrm{i}\beta} + \mathrm{e}^{-\mathrm{i}(\alpha+\beta)}, \quad \alpha, \beta \in [0, 2\pi[. \tag{5.1}$$

If g belongs to a non-generic orbit of type 2 or 3 in Theorem 3.1, then at least two eigenvalues are equal. Thus, setting $\alpha = \beta$ we obtain a curve,

$$[0, 2\pi[\ni \alpha \mapsto T(\alpha) = 2\,\mathrm{e}^{\mathrm{i}\alpha} + \mathrm{e}^{-2\mathrm{i}\alpha} \in \mathbb{C}, \tag{5.2}$$

which turns out to be a hypocycloid, see Fig. 1. We define **D** as the compact region enclosed by this curve. We will show that **D** coincides with the image of the Hilbert mapping ρ. For this purpose we first prove the following

Lemma 5.1. *Any complex number $T \in \mathbb{C}$ can be presented in the following form*:

$$T = s\,\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-2\mathrm{i}\theta}, \tag{5.3}$$

where $s \in \mathbb{R}$, $\theta \in [0, \pi[$.

Proof. It is sufficient to show that the mapping

$$\mathbb{R} \times [0, \pi[\ni (s, \theta) \mapsto \phi(s, \theta) := s\,\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-2\mathrm{i}\theta} \in \mathbb{C}$$

is surjective. Denoting $T = t_1 + \mathrm{i}t_2$ we have:

$$t_1 = s\cos\theta + \cos 2\theta, \qquad t_2 = s\sin\theta - \sin 2\theta. \tag{5.4}$$

We show that for given t_2, t_1 runs over the whole real axis. For $t_2 \neq 0$ ($\sin\theta \neq 0$), we obtain from the second equation in (5.4):

$$s = \frac{t_2 + \sin 2\theta}{\sin\theta}.$$

Substituting this into the first equation of (5.4), we get t_1 as a function of θ:

$$t_1(\theta) = \frac{t_2 + \sin 2\theta}{\sin\theta}\cos\theta + \cos 2\theta.$$

The limits at the boundaries are:

$$\lim_{\theta\to 0^+} t_1(\theta) = \mathrm{sgn}(t_2)\cdot\infty, \qquad \lim_{\theta\to\pi^-} t_1(\theta) = -\mathrm{sgn}(t_2)\cdot\infty.$$

The function $\theta \to t_1(\theta)$ is continuous over the interval $]0,\pi[$, so it takes all mean values. This means that for given $t_2 \neq 0$, $t_1(]0,\pi[) = \mathbb{R}$.

For $t_2 = 0$ we have $\theta = 0$. Then, the first of Eq. (5.4) yields $t_1 = s + 1$. $\square$

Observe that by substituting $(\alpha,\beta) \to (\theta+\phi, \theta-\phi)$ formula (5.1) can be rewritten in the form

$$T(\phi,\theta) = \mathrm{e}^{\mathrm{i}(\theta+\phi)} + \mathrm{e}^{\mathrm{i}(\theta-\phi)} + \mathrm{e}^{-2i\theta},$$

yielding

$$T(\phi,\theta) = (\mathrm{e}^{\mathrm{i}\phi} + \mathrm{e}^{-\mathrm{i}\phi})\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-2\mathrm{i}\theta} = 2\cos\phi\,\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-2\mathrm{i}\theta} = s\,\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-2\mathrm{i}\theta},$$

where we have denoted $s := 2\cos\phi$. Thus, in the parametrization (5.3) we have

$$\mathbf{D} = \{T(s,\theta) = s\,\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-2\mathrm{i}\theta} \in \mathbb{C} : (s,\theta) \in [-2,2]\times[0,\pi[\}$$

and

$$\partial\mathbf{D} = \{T(s,\theta) = s\,\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-2\mathrm{i}\theta} \in \mathbb{C} : \theta \in [0,\pi[,\ s = 2 \text{ or } s = -2\}.$$

But

$$T(-2,\theta) = -2\,\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-2\mathrm{i}\theta} = 2\,\mathrm{e}^{\mathrm{i}(\theta+\pi)} + \mathrm{e}^{-2\mathrm{i}(\theta+\pi)} = T(2,\theta+\pi),$$

and, whence, $\partial\mathbf{D}$ coincides with the hypocycloid defined above,

$$\partial\mathbf{D} = \{T(\theta) = 2\,\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-2\mathrm{i}\theta} \in \mathbb{C} : \theta \in [0,2\pi[\}.$$

One easily checks that in terms of $x = \Re(T)$ and $y = \Im(T)$, $\mathbf{D}$ is given by

$$\mathbf{D} = \{x + \mathrm{i}y \in \mathbb{C} : 27 - x^4 - 2x^2y^2 - y^4 + 8x^3 - 24xy^2 - 18x^2 - 18y^2 \geq 0\}. \tag{5.5}$$

Theorem 5.2. *Let $T \in \mathbb{C}$ and consider the equation*

$$\lambda^3 - T\lambda^2 + \bar{T}\lambda - 1 = 0. \tag{5.6}$$

Its roots $\lambda_1, \lambda_2, \lambda_3$ obey

$$|\lambda_1| = |\lambda_2| = |\lambda_3| = 1, \qquad \lambda_1 + \lambda_2 + \lambda_3 = T, \qquad \lambda_1\lambda_2\lambda_3 = 1, \tag{5.7}$$

if and only if $T \in \mathbf{D}$. Consequently, $\mathrm{tr}(\mathrm{SU}(3)) = \mathbf{D}$.

Proof. Using Lemma 5.1 we can substitute $T(s, \theta) = s\,\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-2\mathrm{i}\theta}$ into Eq. (5.6):

$$\lambda^3 - (s\,\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-2\mathrm{i}\theta})\lambda^2 + (s\,\mathrm{e}^{-\mathrm{i}\theta} + \mathrm{e}^{2\mathrm{i}\theta})\lambda - 1 = 0.$$

It is easy to check that $\lambda_1 = \mathrm{e}^{-2\mathrm{i}\theta}$ is a root of this equation. Thus, we can rewrite it in the form:

$$(\lambda - \mathrm{e}^{-2\mathrm{i}\theta})(\lambda^2 - s\,\mathrm{e}^{\mathrm{i}\theta}\lambda + \mathrm{e}^{2\mathrm{i}\theta}) = 0.$$

Let us find the two remaining solutions. For $|s| \leq 2$ ($T \in \mathbf{D}$) we obtain:

$$\lambda_{2,3} = \frac{s \pm \mathrm{i}\sqrt{4 - s^2}}{2}\mathrm{e}^{\mathrm{i}\theta}, \qquad |\lambda_{2,3}|^2 = \frac{s^2 + 4 - s^2}{4} = 1. \tag{5.8}$$

For $|s| > 2$ we get:

$$\lambda_{2,3} = \frac{s \pm \sqrt{s^2 - 4}}{2}\mathrm{e}^{\mathrm{i}\theta}, \qquad |\lambda_{2,3}|^2 = \left(\frac{s^2 \pm \sqrt{s^2 - 4}}{2}\right)^2 \neq 1.$$

One can check that the sum and the product of roots have the above properties (in both cases).

Finally, recall that the characteristic polynomial of any SU(3)-matrix is of the form (2.9), with eigenvalues uniquely given as roots of this polynomial. Thus, we have shown that the numbers $\{\lambda_1, \lambda_2, \lambda_3\}$ are eigenvalues of the characteristic equation of an SU(3)-matrix g and (5.6) coincides with the characteristic equation of g if and only if $\mathrm{tr}(g) \in \mathbf{D}$, so $\mathrm{tr}(\mathrm{SU}(3)) = \mathbf{D}$. $\square$

To summarize, combining Theorems 3.1 and 5.2 we get the following lemma.

Corollary 5.3. *For $N = 1$, the reduced configuration space $\hat{\mathcal{C}}_\Lambda$ is isomorphic to* $\mathbf{D}$ *and contains three orbit types characterized by the following conditions*:

1. *g has three different eigenvalues* $\Leftrightarrow$ tr g *lies inside* **D**,
2. *g has exactly two different eigenvalues* $\Leftrightarrow$ tr g *lies on the boundary of* **D**, *minus the corners*,
3. $g \in \mathcal{Z} \Leftrightarrow$ tr g *is one of the three corners on the boundary of* **D**.

6. The configuration space for $N = 2$

6.1. Strata in terms of invariants

We define a mapping

$$\rho = (\rho_1 \cdots \rho_9) : \mathbf{G}^2 \longrightarrow \mathbb{R}^9$$

by

$$\rho_1(g,h) := \Re(T_1(g,h)) = \Re(\mathrm{tr}(g)), \tag{6.1}$$

$$\rho_2(g,h) := \Im(T_1(g,h)) = \Im(\mathrm{tr}(g)), \tag{6.2}$$

$$\rho_3(g,h) := \Re(T_2(g,h)) = \Re(\mathrm{tr}(h)), \tag{6.3}$$

$$\rho_4(g,h) := \Im(T_2(g,h)) = \Im(\mathrm{tr}(h)), \tag{6.4}$$

$$\rho_5(g,h) := \Re(T_3(g,h)) = \Re(\mathrm{tr}(hg)), \tag{6.5}$$

$$\rho_6(g,h) := \Im(T_3(g,h)) = \Im(\mathrm{tr}(hg)), \tag{6.6}$$

$$\rho_7(g,h) := \Re(T_4(g,h)) = \Re(\mathrm{tr}(hg)), \tag{6.7}$$

$$\rho_8(g,h) := \Im(T_4(g,h)) = \Im(\mathrm{tr}(hg^2)), \tag{6.8}$$

$$\rho_9(g,h) := \Im(T_5(g,h)) = \Im(\mathrm{tr}(h^2g^2hg) - \mathrm{tr}(h^2ghg^2)). \tag{6.9}$$

By Theorem 4.4, the ρ_i constitute a set of generators of the algebra of invariant polynomials on $\mathbf{G}^2$ with respect to the adjoint action of G. According to [16], the mapping ρ induces a homeomorphism of $X := \mathbf{G}^2/\mathrm{Ad}_G$ onto the image of ρ in $\mathbb{R}^9$. The set $\{\rho_i\}$ of generators is, by Theorem 4.4, subject to a relation, given in Appendix B. We rewrite this relation in terms of the canonical coordinates $\{x_i\}$ on $\mathbb{R}^9$ by substituting

$$\mathrm{tr}(g) = x_1 + \mathrm{i}x_2, \qquad \mathrm{tr}(h) = x_3 + \mathrm{i}x_4,$$

$$\mathrm{tr}(hg) = x_5 + \mathrm{i}x_6, \qquad \mathrm{tr}(hg^2) = x_7 + \mathrm{i}x_8$$

and

$$\Im(\mathrm{tr}(h^2g^2hg) - \mathrm{tr}(h^2ghg^2)) = x_9$$

into its right-hand side. By Lemma 4.3, the resulting polynomial $I_0(x_1, \ldots, x_8)$ is real of order 8 (it is of order 4 in every variable $x_1, \ldots, x_8$). Thus, the relation defines a hypersurface

$Z_1 \subset \mathbb{R}$ of codimension 1 defined by

$$Z_1 := \{(x_1, \dots, x_9) \in \mathbb{R}^9 : I_0(x_1, \dots, x_8) = x_9^2\}$$

and the image $\rho(X)$ is a subset of Z_1. On the other hand, by simple dimension counting we know that X is eight-dimensional. We conclude that there cannot exist further independent relations between generators T_i. Thus, $\rho(X)$ is an eight-dimensional compact subset of Z_1. As already mentioned before, in order to identify $\rho(X)$ explicitly, one has to find a number of inequalities between the above invariants. A full solution of this problem will be presented in a separate paper [22].

Next, let X_i denote the stratum of $\mathbf{G}^2/\mathrm{Ad}_G$ corresponding to orbit type i. We are going to characterize each X_i in terms of the above invariants. We will find a hierarchy of relations: passing from one stratum to a more degenerate one, one has to add some new relations to those which are already fulfilled. This way we obtain a sequence of algebraic surfaces,

$$Z_1 \supset Z_2 \supset Z_3 \supset Z_4 \supset Z_5,$$

characterizing the orbit types. Every Z_i has the property that the image of X_i under the mapping ρ is a subset of Z_i having the dimension of Z_i.

According to Theorem 3.2, a pair (g, h) belongs to a non-generic stratum, i.e., it has orbit type 2 or higher, iff g and h have a common eigenvector. The following lemma is due to Volobuev [23].

Lemma 6.1. *The matrices g and h have a common eigenvector if and only if the following three relations are simultaneously satisfied*:

$$T_5(g, h) = 0, \tag{6.10}$$

$$[g, C + C^{-1}] = [h, C + C^{-1}] = 0, \tag{6.11}$$

where $C := hgh^{-1}g^{-1}$ denotes the group commutator.

Proof. If x is a common eigenvector of g and h then x is an eigenvector of the commutator C with eigenvalue 1. Then the other eigenvalues of C are λ and $\bar{\lambda}$, for some λ obeying $|\lambda|^2 = 1$. In particular, $\mathrm{tr}(C)$ is real. Expressing $\mathrm{tr}(C)$ in terms of generators we obtain

$$\begin{aligned}\mathrm{tr}(hgh^{-1}g^{-1}) = \frac{1}{2}(&|\mathrm{tr}(g)|^2 + |\mathrm{tr}(h)|^2 + |\mathrm{tr}(hg)|^2 + |\mathrm{tr}(hg^2)|^2 + |\mathrm{tr}(g)\mathrm{tr}(hg)|^2 - 3 \\ &+ T_5(g, h)) - \Re(\mathrm{tr}(g)\mathrm{tr}(h)\overline{\mathrm{tr}(hg)}) - \Re(\mathrm{tr}(g)\mathrm{tr}(hg)\overline{\mathrm{tr}(hg^2)}).\end{aligned} \tag{6.12}$$

It follows

$$\Im(\mathrm{tr}(C)) = \frac{1}{2\mathrm{i}} T_5(g, h), \tag{6.13}$$

hence (6.10). Furthermore, the subspace E orthogonal to x is an eigenspace of the Hermitean matrix $C + C^{-1}$ with eigenvalue $\lambda + \bar{\lambda}$. Then $[g, C + C^{-1}]x = 0$ and $[g, C + C^{-1}]E = 0$, hence (6.11). Conversely, assume that (6.10) and (6.11) are satisfied. According to (6.13), then $\mathrm{tr}(C)$ is real. Due to Lemma 5.1, we can write $\mathrm{tr}(C) = s\,\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-2\mathrm{i}\theta}$. The rhs is real iff $s = 2\cos\theta$. Then the reconstruction formula (5.8) for the eigenvalues of C from $\mathrm{tr}(C)$ implies that C has an eigenvalue

$$\lambda_3 = \frac{2\cos\theta - \mathrm{i}\sqrt{4-4\cos^2\theta}}{2}\mathrm{e}^{\mathrm{i}\theta} = (\cos\theta - \mathrm{i}\sin\theta)\,\mathrm{e}^{\mathrm{i}\theta} = 1.$$

If this eigenvalue is degenerate then $C = \mathbb{1}$, i.e., g and h commute and therefore have a common eigenvector (even a common eigenbasis). If the eigenvalue $\lambda_3 = 1$ is non-degenerate then 2 is a non-degenerate eigenvalue of $C + C^{-1}$. Let x be a corresponding eigenvector. According to (6.11),

$$[g, C + C^{-1}]x = 2gx - (C + C^{-1})gx = 0,$$

i.e., gx is again an eigenvector of $C + C^{-1}$ with eigenvalue 2. It follows that x is an eigenvector of g and, similarly, of h. $\square$

In terms of invariants, relation (6.11) can be written as

$$\mathrm{tr}([g, C + C^{-1}] \cdot [g, C + C^{-1}]^\dagger) = 0, \tag{6.14}$$

$$\mathrm{tr}([h, C + C^{-1}] \cdot [h, C + C^{-1}]^\dagger) = 0. \tag{6.15}$$

We omit the lengthy expressions for these equations in terms of generators. We only stress that they do not depend on T_5. Thus, again using the canonical coordinate system, we obtain two polynomials $I_1(x_1, \ldots, x_8)$ and $I_2(x_1, \ldots, x_8)$, which vanish on the non-generic strata:

$$Z_2 := \{(x_1, \ldots, x_9) \in Z_1 : x_9 = 0,\, I_1(x_1, \ldots, x_8) = 0,\, I_2(x_1, \ldots, x_8) = 0\}.$$

The definition of Z_1 implies that condition $x_9 = 0$ is equivalent to $I_0(x_1, \ldots, x_8) = 0$, so Z_2 can be equivalently viewed as a subset of $\mathbb{R}^8$ given by equations $I_0 = 0$, $I_1 = 0$ and $I_2 = 0$. The image of the generic stratum X_1 under the map ρ then is contained in $Z_1 \setminus Z_2$. Hence, inside $\rho(X)$, it is given by the inequalities

$$I_0 > 0 \quad \text{or} \quad I_1 > 0 \quad \text{or} \quad I_2 > 0.$$

One can pass to a set of reduced (with respect to their degree) polynomials $\{I_0, I_1^R, I_2^R\}$,

$$I_1^R := \tfrac{1}{2} I_1 + I_0, \tag{6.16}$$

$$I_2^R := \tfrac{1}{2} I_2 + I_0, \tag{6.17}$$

which generate the same ideal in the polynomial algebra, see Appendix C for their concrete expressions.

The set of orbits of type 3 or higher consists of pairs of commuting matrices. The commutativity of a pair g, h can be expressed in terms of invariants as follows:

$$\mathrm{tr}(hgh^{-1}g^{-1}) - 3 = 0.$$

Taking the imaginary part yields, according to (6.13), $T_5 = 0$. Denoting

$$I_3 = \Re(\mathrm{tr}(hgh^{-1}g^{-1}) - 3),$$

we obtain

$$I_3 = 0.$$

I_3 can be expressed in terms of $T_1, \ldots, T_4$, and in terms of canonical coordinates it takes the form

$$\begin{aligned} I_3(x_1, \ldots, x_8) = {} & x_1^2 x_5^2 + x_1^2 x_6^2 + x_2^2 x_5^2 + x_2^2 x_6^2 - 2x_1x_5x_7 - 2x_1x_5x_3 - 2x_1x_6x_8 \\ & - 2x_1x_6x_4 - 2x_2x_5x_8 + 2x_2x_5x_4 + 2x_2x_6x_7 - 2x_2x_6x_3 + x_1^2 \\ & + x_2^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2 + x_3^2 + x_4^2 - 9. \end{aligned}$$

Then, the image of the stratum X_3 under the mapping ρ is a subset of

$$Z_3 := \{(x_1, \ldots, x_9) \in Z_2 : I_3(x_1, \ldots, x_8) = 0\}.$$

Since $\Re(\mathrm{tr}(hgh^{-1}g^{-1}) - 3) \leq 0$, the image of the stratum X_2 under ρ is given, as a subset of $\rho(X)$, by the following equations and inequalities

$$I_0 = 0, \qquad I_1 = 0, \qquad I_2 = 0, \qquad I_3 < 0.$$

The set of orbits of type 4 or higher consists of commuting pairs with a common two-dimensional eigenspace. This implies that both matrices and all their products have degenerate eigenvalues. The invariants T_i, $i = 1, \ldots, 4$, are trace functions of products of SU(3)-matrices, so they take values in $\mathbf{D}$, see Theorem 5.2. Thus, by Corollary 5.3, the values of all invariants T_i, $i = 1, \ldots, 4$, computed on degenerate elements have to belong to $\partial\mathbf{D}$. The polynomial defining this boundary has the following form, see (5.5):

$$B(x_1, x_2) := 27 - x_1^4 - 2x_1^2 x_2^2 - x_2^4 + 8x_1^3 - 24x_1 x_2^2 - 18x_1^2 - 18x_2^2.$$

Thus, we have

$$Z_4 := \{(x_1, \ldots, x_9) \in Z_3 : B(x_1, x_2) = B(x_3, x_4) = B(x_5, x_6) = B(x_7, x_8) = 0\}.$$

Accordingly, the image of the stratum X_3 under the map ρ is given, as a subset of $\rho(X)$, by the relations

$$I_0 = 0, \qquad I_1 = 0, \qquad I_2 = 0, \qquad I_3 = 0$$

and the inequalities

$$B(x_1, x_2) > 0 \quad \text{or} \quad B(x_3, x_4) > 0 \quad \text{or} \quad B(x_5, x_6) > 0 \quad \text{or} \quad B(x_7, x_8) > 0.$$

Finally, the subset of orbits of type 5 consists of pairs of matrices belonging to $\mathcal{Z}$. They fulfill $|\mathrm{tr}(g)| = |\mathrm{tr}(h)| = 3$, so we have

$$Z_5 := \{(x_1, \ldots, x_9) \in Z_4 : x_1^2 + x_2^2 - 9 = 0, x_3^2 + x_4^2 - 9 = 0\}$$

and the image of the stratum X_4 under the map ρ is given, as a subset of $\rho(X)$, by

$$I_0 = I_1 = I_2 = I_3 = B(x_1, x_2) = B(x_3, x_4) = B(x_5, x_6) = B(x_7, x_8) = 0$$

and

$$x_1^2 + x_2^2 - 9 < 0 \quad \text{or} \quad x_3^2 + x_4^2 - 9 < 0.$$

6.2. *Geometric structure of strata*

In this section we give a description of the strata in terms of subsets and quotients of $G = \mathrm{SU}(3)$ and calculate their dimensions. We use the following notation. Let H be a subgroup of G. Then

$N(H) :=$ normalizer of H in G, $\quad \mathbf{G}^2_H :=$ set of pairs (g, h) with stabilizer H,

$\mathbf{G}^2_{(H)} :=$ set of pairs (g, h) invariant under H,

$\mathbf{G}^2_{[H]} :=$ set of pairs (g, h) of type $[H]$.

We obviously have $\mathbf{G}^2_H \subset \mathbf{G}^2_{(H)}$ and $\mathbf{G}^2_H \subset \mathbf{G}^2_{[H]}$. Since we have labelled the orbit types $[H]$ by $i = 1, \ldots, 5$, we denote the strata $\mathbf{G}^2_{[H]}$ by $\mathbf{G}^2_i$. Moreover, in what follows, the symbol $\setminus$ always means taking the set theoretical complement, whereas $/$ means taking the quotient.

For orbit type 5, Theorem 3.2 immediately yields that the corresponding stratum is

$$X_5 = \mathcal{Z} \times \mathcal{Z}.$$

It consists of nine isolated points.

For the remaining orbit types, recall from the basic theory of Lie group actions [17] that the projection $\pi_i : \mathbf{G}^2_i \to X_i$ is a locally trivial fibre bundle with typical fibre G/H associated with the $N(H)/H$-principal bundle $\mathbf{G}^2_H \to X_i$, which is naturally embedded into the associated bundle. Here H is a representative of the conjugacy class i and we have the following diffeomorphism

$$X_i \cong \mathbf{G}^2_H / N(H)/H, \tag{6.18}$$

where $N(H)/H$ is the right coset group acting by inner automorphisms on $\mathbf{G}^2_H$. Thus, for each orbit type we have to choose a representative and then compute the rhs of (6.18).

We start with orbit type 4. As a representative, we choose the subgroup (3.2). Let us denote it by $U(2)_1$. We have

$$\mathbf{G}^2_{U(2)_1} = \mathbf{G}^2_{(U(2)_1)} \setminus \mathcal{Z} \times \mathcal{Z}$$

and

$$\mathbf{G}^2_{(U(2)_1)} = C(U(2)_1) \times C(U(2)_1) = U(1)_1 \times U(1)_1, \tag{6.19}$$

where $C(\cdot)$ denotes the centralizer in G and $U(1)_1$ denotes the subgroup (3.3). Hence,

$$\mathbf{G}^2_{U(2)_1} = U(1)_1 \times U(1)_1 \backslash \mathcal{Z} \times \mathcal{Z}.$$

Since $U(2)_1$ and $U(1)_1$ centralize each other, their normalizers coincide. Since the only way in which $N(U(1)_1)$ can act on $U(1)_1$ is by a permutation of the entries, it must act trivially. It follows

$$N(U(2)_1) = N(U(1)_1) = C(U(1)_1) = U(2)_1,$$

and the factorization in (6.18) is trivial. Therefore, (6.18) yields

$$X_4 \cong U(1)_1 \times U(1)_1 \backslash \mathcal{Z} \times \mathcal{Z}.$$

The dimension of X_4 is 2.

As a representative for orbit type 3 we choose the subgroup (3.1) of diagonal matrices. Let us denote it by T. The set $\mathbf{G}^2_T$ consists of the pairs that are invariant under T minus those that are of orbit type 4 or higher, i.e., that are conjugate to a pair invariant under $U(2)_1$:

$$\mathbf{G}^2_T = \mathbf{G}^2_{(T)} \backslash \left(\bigcup_{g \in G} g\, \mathbf{G}^2_{(U(2)_1)}\, g^{-1} \right).$$

We have

$$\mathbf{G}^2_{(T)} = C(T) \times C(T) = T \times T \tag{6.20}$$

and, from formula (6.19),

$$g\mathbf{G}^2_{(U(2)_1)} g^{-1} = g(U(1)_1 \times U(1)_1) g^{-1} = (gU(1)_1 g^{-1}) \times (gU(1)_1 g^{-1}).$$

Subtraction of this subset from $T \times T$ is only non-trivial if $gU(1)_1 g^{-1} \subseteq T$. The subgroups arising this way are $U(1)_1$ as well as

$$U(1)_2 = \{\mathrm{diag}(\beta, \alpha, \beta) : \alpha, \beta \in U(1), \beta^2 = \bar{\alpha}\},$$
$$U(1)_3 = \{\mathrm{diag}(\beta, \beta, \alpha) : \alpha, \beta \in U(1), \beta^2 = \bar{\alpha}\}.$$

Thus,

$$\mathbf{G}^2_T = T \times T \backslash \left(\bigcup_{i=1}^{3} U(1)_i \times U(1)_i \right).$$

The quotient $N(T)/T$ is the Weyl group of $G = \mathrm{SU}(3)$, isomorphic to the permutation group S_3. Hence,

$$X_3 = \left(T \times T \setminus \left(\bigcup_{i=1}^{3} U(1)_i \times U(1)_i \right) \right) / S_3,$$

where S_3 acts on the elements of T by permuting the entries. The dimension of the stratum X_3 is 4. Note that if we take the quotient $(T \times T)/S_3$, also the points of orbit type 4 and 5 are factorized in the proper way. One can make this precise by saying that $(T \times T)/S_3$ is isomorphic, as a stratified space, to the subspace

$$X_3 \cup X_4 \cup X_5 \subseteq X = \mathbf{G}^2/\mathrm{Ad}_G.$$

As we will see below, this is not true in general.

Next, consider orbit type 2. As a representative, we choose the subgroup $U(1)_1$, given by (3.3). Using an argument analogous to that for orbit type 3, together with formula (6.20) and $C(U(1)_1) = U(2)_1$, we find

$$\mathbf{G}^2_{U(1)_1} = U(2)_1 \times U(2)_1 \setminus \left(\bigcup_{g \in G} g(T \times T)g^{-1} \right).$$

A pair $(g, h) \in U(2)_1 \times U(2)_1$ is conjugate to an element of $T \times T$ iff g and h belong to the same maximal toral subgroup in $U(2)_1$. Thus,

$$\mathbf{G}^2_{U(1)_1} = U(2)_1 \times U(2)_1 \setminus \left(\bigcup_{\widetilde{T}} \widetilde{T} \times \widetilde{T} \right),$$

where the union is over all maximal tori in $U(2)_1$. As for the normalizer, we already know that $N(U(1)_1) = U(2)_1$, hence we have to factorize by $U(2)_1/U(1)_1 \cong \mathrm{SU}(2)$, i.e., by $U(2)_1$ modulo its center:

$$X_2 \cong \left(U(2)_1 \times U(2)_1 \setminus \left(\bigcup_{\widetilde{T}} \widetilde{T} \times \widetilde{T} \right) \right) / U(2)_1/U(1)_1.$$

We see that this stratum has dimension 5. We remark that in (6.21) it is important to remove the pairs of higher symmetry, because they would not be factorized in the proper way here. Since $U(1)_1$ is the center of $U(2)_1$, we get

$$X_2 \cong \left(U(2)_1 \times U(2)_1 \setminus \left(\bigcup_{\widetilde{T}} \widetilde{T} \times \widetilde{T} \right) \right) / U(2)_1. \tag{6.21}$$

Moreover, $\bigcup_{\widetilde{T}} \widetilde{T} \times \widetilde{T}$ contains all non-generic orbit types of the $U(2)_1$-action. Hence, the rhs of (6.21) is isomorphic to the generic stratum of the orbit space of the action of the abstract Lie group $U(2)$ by diagonal conjugation on $U(2) \times U(2)$, i.e.,

$$X_2 \cong ((U(2) \times U(2))/U(2))_{\mathrm{gen}}. \tag{6.22}$$

One option to analyze this quotient is to restrict the action to the subgroup $\mathrm{SU}(2) \subset U(2)$ and to rewrite the two factors $U(2)$ using the Lie group isomorphism

$$U(2) \cong (U(1) \times \mathrm{SU}(2))/\mathbb{Z}_2,$$

thus obtaining

$$(U(2) \times U(2))/U(2) \cong (U(1) \times U(1) \times ((\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathrm{SU}(2)))/(\mathbb{Z}_2 \times \mathbb{Z}_2).$$

Here the quotient $(\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathrm{SU}(2)$ is known as the "pillow". It consists of a three-dimensional stratum (corresponding to the interior), a two-dimensional stratum (the boundary minus the four edges) and a zero-dimensional stratum (the four edges).

Another option is to apply an algorithm which provides a decomposition of quotients of diagonal (or joint) actions on direct product spaces into quotients of the individual factors. Since we will use this algorithm again to describe the generic stratum X_1 below, we will explain it in some generality. Let H be a Lie group acting on a manifold M and consider the diagonal action of H on $M \times M$ (one can easily generalize the procedure to diagonal action on $M_1 \times \cdots \times M_n$). In what follows, we denote the sets of orbit types of the action of H on M, of a subgroup $K \subseteq H$ on M and of H on $M \times M$ by $\mathcal{O}(M, H)$, $\mathcal{O}(M, K)$ and $\mathcal{O}(M \times M, H)$, respectively. We start with decomposing

$$(M \times M)/H = \bigcup_{[K] \in \mathcal{O}(M,H)} (M_{[K]} \times M)/H.$$

If two pairs $(x_1, x_2), (y_1, y_2) \in M_K \times M \subset M_{[K]} \times M$ are conjugate under $h \in H$, then conjugation of the stabilizer of x_1 by h yields the stabilizer of y_1. Since both are equal to K, h is in the normalizer of K in H, $h \in N(K)$. Thus,

$$(M_{[K]} \times M)/H = (M_K \times M)/N(K),$$

for some fixed representative K of the orbit type $[K]$. Factorization by $N(K)$ can be achieved by first factorizing by K and then by $N(K)/K$. Since K acts trivially on the factor M_K, we obtain

$$(M \times M)/H = \bigcup_{[K] \in \mathcal{O}(M,H)} (M_K \times (M/K))/N(K)/K. \tag{6.23}$$

We decompose M/K by orbit types of the K-action on M:

$$M/K = \bigcup_{[K']_K \in \mathcal{O}(M,K)} (M/K)_{[K']_K}. \tag{6.24}$$

Here $[K']_K$ denotes the conjugacy class of the subgroup $K' \subseteq K$ in K. Inserting (6.24) into (6.23), we obtain

$$(M \times M)/H = \bigcup_{[K]\in\mathcal{O}(M,H)} \left(M_K \times \left(\bigcup_{[K']_K \in\mathcal{O}(M,K)} (M/K)_{[K']_K} \right) \right) /N(K)/K. \tag{6.25}$$

Consider, on the other hand, the decomposition of $(M \times M)/H$ by orbit types,

$$(M \times M)/H = \bigcup_{[L]\in\mathcal{O}(M\times M,H)} ((M \times M)/H)_{[L]}.$$

A representative of the rhs of (6.25) is given by (x, y), where $x \in M_K$ and y can be chosen such that it has orbit type K' under the action of K. The stabilizer of this pair under the action of H is given by intersecting the stabilizer of x under the action of H, which is K, with the stabilizer of y under the action of H. The intersection yields the stabilizer of y under the action of K, which is K'. Hence, the stabilizer of (x, y) under the action of H is K' and the orbit type is $[K']$, where the conjugacy class is taken in H. Thus, for every $[L] \in \mathcal{O}(M \times M, H)$, we have

$$((M \times M)/H)_{[L]} = \bigcup_{[K]\in\mathcal{O}(M,H)} \left(M_K \times \left(\bigcup_{\substack{[K']_K \in\mathcal{O}(M,K)\\ [K']=[L]}} (M/K)_{[K']_K} \right) \right) /N(K)/K. \tag{6.26}$$

At this stage, the equality sign just means bijective correspondence on the level of abstract sets. Of course, this can be made more precise by saying how the individual manifolds on the rhs are glued together to build up the manifold on the lhs. Here we do not elaborate on this, for details we refer to [22].

Let us apply (6.26) to the quotient given by (6.22), i.e. to the case $M = H = U(2)$ with conjugate action. Representatives of orbit types of the $U(2)$-action on $U(2)$ are $K = U(2)$ and $K = T$, where T denotes the subgroup of $U(2)$ consisting of diagonal matrices (obviously, if we identify $U(2)$ with the subgroup $U(2)_1$ of SU(3), this is consistent with the notation T used above). Representatives of orbit types of the K-action on $U(2)$ are $K' = U(2)$, T for $K = U(2)$ and $K' = T$, $U(1)$ for $K = T$. Here $U(1)$ denotes the center of $U(2)$. Hence, the only piece in the decomposition (6.26) that belongs to the generic stratum of the $U(2)$-action on $U(2) \times U(2)$ (orbit type $[U(1)]$) is that labelled by the subgroups $K = T$ and $K' = U(1)$. The first factor of this piece is

$$U(2)_T = T \setminus U(1),$$

the second one

$$(U(2)/T)_{[U(1)]_T} = (U(2)/T)_{\mathrm{gen}}.$$

The quotient group $N(K)/K = N(T)/T$ is the Weyl group of $U(2)$. It is isomorphic to the permutation group S_2 and can be represented on $U(2)$ by conjugation by the permutation matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Of course, on the first factor this amounts to interchanging the entries. Thus, we end up with

$$X_2 \cong ((U(2) \times U(2))/U(2))_{\text{gen}} = ((T \setminus U(1)) \times \big(U(2)/T)_{\text{gen}})/S_2.$$

Clearly, $(U(2)/T)_{\text{gen}}$ can be further analyzed, in a similar way as above.

Finally, consider the generic stratum X_1. Again, we apply (6.26), where now $M = H = G = \text{SU}(3)$ with conjugate SU(3)-action. Representatives of orbit types of the G-action on G are $K = G$, $U(2)_1$, and T. For $K = G$, the orbit types of the K-action on G are again $[G]$, $[U(2)]$ and $[T]$, hence these pieces do not contribute to X_1. For $K = U(2)_1$ and $K = T$, the K-action on G has one orbit type represented by $\mathcal{Z}$. For both actions, this orbit type is the generic one. Thus, for X_1, the decomposition (6.26) consists of one piece labelled by the subgroups $K = U(2)_1$ and $K' = \mathcal{Z}$ and one piece labelled by $K = T$ and $K' = \mathcal{Z}$. Computing these pieces we obtain

$$X_1 = (U(1)_1 \setminus \mathcal{Z}) \times (G/U(2)_1)_{\text{gen}} \cup \left(\left(T \setminus \left(\bigcup_{i=1}^{3} U(1)_i \right) \right) \times (G/T)_{\text{gen}} \right) / S_3,$$

where the action of the Weyl group S_3 on G can be represented by conjugation by the 3×3-permutation matrices. These are generated, e.g., by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(Notice that the permutation matrices of negative sign have determinant -1, hence they are not in SU(3).) On the first factor, S_3 acts by permuting the entries. We note again that the quotients $(G/U(2)_1)_{\text{gen}}$ and $(G/T)_{\text{gen}}$ can be further analyzed.

6.3. Representatives of orbits

As above, we denote strata by $\mathbf{G}_i^2 \subset \mathbf{G}^2$, and the corresponding pieces of the stratified orbit space by $X_i = \mathbf{G}_i^2/\text{Ad}_G \subset \mathbf{G}^2/\text{Ad}_G$, $i = 1, \ldots, 5$. In this subsection we present representatives for each orbit type. More precisely, we define *local* cross sections

$$X_i \supset \mathcal{U}_i \ni [\mathbf{g}] \to \mathbf{s}([\mathbf{g}]) \equiv (s_1, s_2)([\mathbf{g}]) \in \mathbf{G}_i^2,$$

for each bundle

$$\pi_i : \mathbf{G}_i^2 \to X_i.$$

Here, $\mathcal{U}_i$ denotes a dense subset of X_i. For that purpose, we use a system of local trivializations of SU(3), viewed as an SU(2)-principal bundle over S^5, see Appendix A.

6.3.1. The generic stratum

The projection $\pi_1 : \mathbf{G}_1^2 \to X_1$ of the generic stratum is a locally trivial principal fibre bundle with structure group $G/\mathcal{Z}$. Using arguments developed in [24] one can prove that this bundle is non-trivial and that one can find a system of local trivializations (respectively local cross sections), defined over a covering of X_1 with open subsets, which are all dense with respect to the natural measure (the one induced by the Haar-measure).

Proposition 6.2. *There exists a local cross section*

$$X_1 \supset \mathcal{U}_1 \ni [\mathbf{g}] \to \mathbf{s}([\mathbf{g}]) \equiv (s_1, s_2)([\mathbf{g}]) \in \mathbf{G}_1^2,$$

of the generic stratum with $\mathbf{s}$ *given by*

$$s_1 = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \qquad s_2 = \left[\begin{array}{c|c} a & -\delta^{-1} b^\dagger \\ \hline b & \delta\left(\mathbb{1} - \frac{bb^\dagger}{1+|a|}\right) \end{array}\right] \times \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & c & d \\ 0 & -\bar{d} & \bar{c} \end{array}\right], \tag{6.27}$$

where:

$$|\lambda_1| = |\lambda_2| = |\lambda_3| = 1, \qquad \lambda_1\lambda_2\lambda_3 = 1, \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \qquad b_1, b_2 \in \mathbb{R}_+,$$
$$|a|^2 + b_1^2 + b_2^2 = 1, \qquad a = |a|\delta^{-2}, \qquad |c|^2 + |d|^2 = 1. \tag{6.28}$$

Proof. Let

$$X_1 \supset \mathcal{U}_1 \ni [\mathbf{g}] \to \mathbf{s}([\mathbf{g}]) \equiv (s_1, s_2)([\mathbf{g}]) \in \mathbf{G}_1^2$$

be a local cross section, with $\mathcal{U}_1$ dense in X_1. Since Ad_G acts (pointwise) on this cross section, we can fix the gauge by bringing $\mathbf{s}$ to a special form. Since s_1 and s_2 are in generic position on $\mathcal{U}_1$, they have no common eigenvector and at least one element of this pair, say s_1, has three different eigenvalues. Thus, on this neighborhood, we can fix the gauge in two steps: first, we diagonalize s_1 and next we use the stabilizer of this diagonal element to bring s_2 to a special form. Since s_1 and s_2 have no common eigenvector, this fixes the (remaining) stabilizer gauge completely (up to $\mathbb{Z}_3$). Thus, we can assume that s_1 is diagonal, with eigenvalues ordered in a unique way, and that s_2 has the form, defined by the cross section (A.13) in Appendix A,

$$s_2 = \left[\begin{array}{c|c} a & -\delta_i b^\dagger \\ \hline b & \delta_i^{-1}\left(\mathbf{1} - \frac{bb^\dagger}{1+|a|}\right) \end{array}\right] \times \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & S \end{array}\right], \qquad S \in \mathrm{SU}(2). \tag{6.29}$$

Let

$$\pi_1^{-1}(\mathcal{U}_1) \ni (s_1, s_2) \mapsto f(s_1, s_2) \in G$$

belong to the stabilizer of s_1. Since s_1 is diagonal, f can be written in the form

$$f = \begin{bmatrix} \mathrm{e}^{-\mathrm{i}(\alpha+\beta)} & 0 & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\alpha} & 0 \\ 0 & 0 & \mathrm{e}^{\mathrm{i}\beta} \end{bmatrix}.$$

The action of f on an arbitrary group element g is given by:

$$\begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \to \begin{bmatrix} g_{11} & \mathrm{e}^{-\mathrm{i}(\alpha+2\beta)} g_{12} & \mathrm{e}^{-\mathrm{i}(2\alpha+\beta)} g_{13} \\ \mathrm{e}^{\mathrm{i}(\alpha+2\beta)} g_{21} & g_{22} & \mathrm{e}^{-\mathrm{i}(\beta-\alpha)} g_{23} \\ \mathrm{e}^{\mathrm{i}(2\alpha+\beta)} g_{31} & \mathrm{e}^{\mathrm{i}(\beta-\alpha)} g_{32} & g_{33} \end{bmatrix}. \tag{6.30}$$

Thus, we can choose the phases α and β in such a way that after transformation with f, the entries b_i of b occurring in (6.29) are real and positive. □

By the results of Section 6.1, it is clear that the representative $\mathbf{s}$ can be expressed in terms of invariants $t_i := T_i(s_1, s_2)$, $i = 1, \dots, 5$. With some effort, one can find these expressions explicitly. Here, we only sketch how to do that. In Section 5 we have already found the eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$ in terms of $t_1 = \mathrm{tr}(s_1)$. Thus, we are left with calculating s_2. For that purpose, denote the diagonal entries of s_2 by x, y and z. Then, we have

$$t_2 = x + y + z, \qquad t_3 = \lambda_1 x + \lambda_2 y + \lambda_3 z, \qquad t_4 = \lambda_1^2 x + \lambda_2^2 y + \lambda_3^2 z.$$

This is system of linear equations for x, y, z, which can be trivially solved. The second, non-trivial step consists in expressing the parameters a, b, c, d, δ in terms of x, y, z, by solving the set of non-linear equations

$$x = a, \qquad y = \delta c - \frac{\delta}{1+|a|}(b_1^2 c - b_1 b_2 \bar{d}),$$
$$z = \delta \bar{c} - \frac{\delta}{1+|a|}(b_1 b_2 d + b_2^2 \bar{c}), \tag{6.31}$$

where, of course, relations (6.28) have to be taken into account. It can be shown that this set of equations has two solutions, corresponding to different parameters b_1, b_2, d:

$$a = x, \qquad \delta = \sqrt{\frac{|a|}{a}}, \qquad c = \frac{\bar{\delta} y + \delta \bar{z}}{1 + |a|},$$

$$b_1^{\pm} = \frac{1}{\sqrt{2}}[2(c_1 q_1 + c_2 q_2) + (1 - |c|^2)(1 - |a|^2) \pm \sqrt{\Delta}]^{1/2},$$

$$b_2^{\pm} = \sqrt{1 - |a|^2 - b_1^2}, \qquad d_1^{\pm} = \frac{c_1 b_1^2 - q_1}{b_1 b_2}, \qquad d_2^{\pm} = \frac{-c_2 b_1^2 + q_2}{b_1 b_2},$$

where

$$c_1 := \mathrm{Re}(c), \qquad c_2 := \mathrm{Im}(c), \qquad d_1 := \mathrm{Re}(d), \qquad d_2 := \mathrm{Im}(d),$$
$$q_1 := -\mathrm{Re}(\bar{\delta} y - c)(1 + |a|), \qquad q_2 := -\mathrm{Im}(\bar{\delta} y - c)(1 + |a|),$$

and

$$\Delta = [2(c_1 q_1 + c_2 q_2) + (1 - |c|^2)(1 - |a|^2)]^2 - 4(q_1^2 + q_2^2).$$

Next, observe that the matrices described by these two sets of parameters are related, namely one of them is equal to the transposition of the second one. On the other hand, all invariants t_i, $i = 1, \ldots, 4$, are invariant under transposition of matrices. The two solutions are distinguished by the value of $T_5(s_1, s_2)$, which has the property

$$T_5(s_1, s_2) = -T_5(s_1^{\mathrm{T}}, s_2^{\mathrm{T}}).$$

In terms of matrix elements of s_1 and s_2, T_5 has the following form:

$$T_5(s_1, s_2) = \pm(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1)\sqrt{\Delta}.$$

Thus, calculating the value of $T_5(s_1, s_2)$ enables us to choose the correct sign in front of the square root of Δ and to obtain a unique solution.

6.3.2. The $U(1)$-stratum

Let $\mathbf{s}$ be a local cross section of the (non-trivial) bundle $\pi_2 : \mathbf{G}_2^2 \to X_2$. There exists one common eigenvector of s_1 and s_2. Assume that it is the first eigenvector of s_1. After diagonalizing s_1, the pair (s_1, s_2) has the following form

$$s_1 = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \qquad s_2 = \left[\begin{array}{c|c} \det(S)^{-1} & 0 \\ \hline 0 & S \end{array}\right], \tag{6.32}$$

where $S \in U(2)$. The stabilizer $H_{\mathbf{s}} \cong U(1)$ of $\mathbf{s}$ is given by (3.3). Thus, to obtain a cross section, we have to fix the S_2-action, which permutes the second and third basis vectors and the $H_{\mathbf{s}}$-action on s_2. First, since $\lambda_2 \neq \lambda_3$, these eigenvalues can be uniquely ordered, for example by increasing phase. Next, the $H_{\mathbf{s}}$-action is fixed by requiring that the left lower entry of s_2 has to be real and positive. Thus, we get the following local cross section:

$$s_1 = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \qquad s_2 = \left[\begin{array}{c|cc} \delta^{-2} & 0 & 0 \\ \hline 0 & \delta c & -\delta^2 d \\ 0 & d & \delta \bar{c} \end{array}\right], \tag{6.33}$$

where:

$$|\lambda_1| = |\lambda_2| = |\lambda_3| = 1, \qquad \lambda_1\lambda_2\lambda_3 = 1, \qquad |\delta| = 1,$$
$$|c|^2 + d^2 = 1, \qquad d \in \mathbb{R}_+.$$

Again, the representative (6.33) can be expressed in terms of invariants: the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of s_1 are given in terms of t_1. If $\lambda_1 \neq \lambda_2$, we can proceed in the same way as for the generic stratum above, i.e., by solving the set of Eqs. (6.31). This way, we obtain the diagonal components δ^{-2}, δc, $\delta \bar{c}$ of s_2, and we can compute the coefficients c and δ. There exist two solutions for c and δ but they describe the same matrix. If $\lambda_1 = \lambda_2$, Eq. (6.31) imply

$$(\delta^{-2} + \delta c) = (x + y), \qquad \delta \bar{c} = z,$$

which can be solved with respect to c and δ^2:

$$\delta^2 = \frac{2}{(x+y) \pm \sqrt{(x+y)^2 - 4\bar{z}}}, \qquad c = \delta \bar{z}.$$

(There are two values for δ^2, but only one of them satisfies the condition $|\delta|^2 = 1$. Taking the square root of the correct one then yields two solutions for δ, but these give the same matrix.) Finally, one calculates

$$d = \sqrt{1 - |c|^2}.$$

6.3.3. The $U(1) \times U(1)$-stratum

Let $\mathbf{s}$ be a local cross section of the (non-trivial) bundle $\pi_3 : \mathbf{G}_3^2 \to X_3$. In this case, s_1 and s_2 can be jointly diagonalized:

$$s_1 = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \qquad s_2 = \begin{bmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_3 \end{bmatrix},$$

where

$$|\lambda_1| = |\lambda_2| = |\lambda_3| = 1, \qquad \lambda_1\lambda_2\lambda_3 = 1,$$
$$|\delta_1| = |\delta_2| = |\delta_3| = 1, \qquad \delta_1\delta_2\delta_3 = 1.$$

Since there is no common two-dimensional eigenspace, the remainder of the action of the stabilizer $H_{\mathbf{s}} \cong U(1) \times U(1)$ is the permutation group S_3. To fix the S_3-action, observe that, according to Corollary 3.3, either one of the matrices has three different eigenvalues or both have a pair of degenerate eigenvalues corresponding to distinct eigenspaces. In the first case, we can fix the S_3-action by ordering the three distinct eigenvalues. In the second case, we can put the unique non-degenerate eigenvalue of s_1 in the first place and establish the order of the two remaining eigenvectors by ordering the corresponding two distinct eigenvalues of s_2.

Expressing $\mathbf{s}$ in terms of invariants is then immediate: all eigenvalues can be calculated in terms of the traces $t_1 = \mathrm{tr}(s_1)$ and $t_2 = \mathrm{tr}(s_2)$. To determine which eigenvalues of s_1 and s_2 correspond to the same eigenvector it is sufficient to know the value of $t_3 = \mathrm{tr}(s_1 s_2)$. It can take six values corresponding to the permutations of the eigenvalues of s_2 relative to those of s_1.

6.3.4. The $U(2)$-stratum

Let $\mathbf{s}$ be a cross section of the (trivial) bundle $\pi_4 : \mathbf{G}_4^2 \to X_4$. Obviously, $\mathbf{s}$ can be taken in the following form:

$$s_1 = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}, \qquad s_2 = \begin{bmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_2 \end{bmatrix},$$

where $|\lambda_1| = |\lambda_2| = |\delta_1| = |\delta_2| = 1$, $\lambda_1 \lambda_2^2 = \delta_1 \delta_2^2 = 1$. For expressing (s_1, s_2) in terms of invariants it is sufficient to know the values t_1 and t_2, because there is only one possible order.

6.3.5. The SU(3)*-stratum*

Let $\mathbf{s}$ be a cross section of the (trivial) bundle $\pi_5 : \mathbf{G}_5^2 \to X_5$. Then,

$$s_1 = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}, \qquad s_2 = \begin{bmatrix} \delta & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \delta \end{bmatrix},$$

define a unique cross section, with $\lambda^3 = 1$ and $\delta^3 = 1$. The traces of both matrices take one of the following three values: $3\,\mathrm{e}^{\mathrm{i}(2k\pi/3)}$, $k = 0, 1, 2$. Thus, expressing them in terms of invariants is trivial.

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Appendix A. A principal bundle atlas for the SU(3) group manifold

It is well known that the group SU(3) can be viewed as a principal bundle over the sphere S^5 with structure group SU(2),

$$\mathrm{SU}(2) \hookrightarrow \mathrm{SU}(3) \xrightarrow{\pi} S^5, \tag{A.1}$$

with π being the canonical projection from SU(3) onto the right coset space $\mathrm{SU}(3)/\mathrm{SU}(2) \cong S^5$. An explicit description of π is obtained as follows: any 3×3 matrix can be written in the form

$$g = \left[\begin{array}{c|c} a & c^\dagger \\ \hline b & B \end{array}\right], \tag{A.2}$$

with $a \in \mathbb{C}$, $b, c \in \mathbb{C}^2$ and a complex 2×2-matrix B. The condition that g belongs to $U(3)$, namely

$$gg^\dagger = \mathbf{1} = g^\dagger g,$$

translates into the following relations for entries of g:

$$|a|^2 + \|b\|^2 = 1 = |a|^2 + \|c\|^2, \tag{A.3}$$

$$\bar{a}b + Bc = 0 = ac + B^\dagger b, \tag{A.4}$$

$$bb^\dagger + BB^\dagger = \mathbf{1} = cc^\dagger + B^\dagger B. \tag{A.5}$$

We embed the subgroup SU(2) of SU(3) as follows:

$$\mathrm{SU}(2) \ni S \to h = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & S \end{array}\right] \in \mathrm{SU}(3).$$

Observe that then SU(2) is the stabilizer of the vector

$$e_1 := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in S^5 \subset \mathbb{C}^3.$$

The image of the left action of $g \in \mathrm{SU}(3)$ on e_1 is exactly the first column of g, which – on the other hand – is also invariant under right action of SU(2). Thus, $\pi(g)$ can be identified with the first column of g,

$$\pi(g) = \begin{bmatrix} a \\ b_1 \\ b_2 \end{bmatrix} \in S^5 \subset \mathbb{C}^3,$$

which by (A.3) has norm 1, indeed.

Next, we construct an atlas of local trivializations of the bundle (A.1). Observe first that, according to (A.5), $\det(B) = 0$ iff $\|b\| = 1$ and, whence, iff $a = 0$. Thus, let us assume $a \neq 0$ and construct appropriate trivializations of (A.1) over the open set $\mathcal{O} = \{(a, b)|a \neq 0\} \subset S^5$. Using the polar decomposition $B = AV$, where $A > 0$, $V \in U(2)$, we can rewrite Eq. (A.5) as follows:

$$bb^\dagger = \mathbf{1} - A^2 = Vcc^\dagger V^\dagger,$$

yielding

$$c = -\mathrm{e}^{-\mathrm{i}\phi} V^\dagger b, \quad \phi \in \mathbb{R}, \tag{A.6}$$

$$A^2 = \mathbf{1} - bb^\dagger. \tag{A.7}$$

Formulae (A.4) and (A.6) imply

$$(\bar{a} - \mathrm{e}^{-\mathrm{i}\phi} A)b = 0,$$

which means that b is an eigenvector of the matrix A with eigenvalue $\bar{a}\,\mathrm{e}^{\mathrm{i}\phi}$. Positivity of A implies $|a| = \bar{a}\,\mathrm{e}^{\mathrm{i}\phi}$.

From Eq. (A.7) we have $A = \sqrt{\mathbf{1} - bb^\dagger}$. Since $A > 0$ this formula defines A uniquely. Obviously, it must be of the form

$$A = \alpha\mathbf{1} + \beta bb^\dagger. \tag{A.8}$$

Plugging this into Eq. (A.7) yields

$$A = \mathbf{1} - \frac{1}{1 + |a|} bb^\dagger. \tag{A.9}$$

We conclude that any matrix $g \in U(3)$ which fulfils the condition $a \neq 0$ can be written in the following form:

$$g = \left[\begin{array}{c|c} a & -\mathrm{e}^{\mathrm{i}\phi} b^\dagger \\ \hline b & \mathbf{1} - \frac{bb^\dagger}{1+|a|} \end{array}\right] \cdot \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & V \end{array}\right], \tag{A.10}$$

with $|a|^2 + \|b\|^2 = 1, \quad a = |a|\,\mathrm{e}^{\mathrm{i}\phi}, V \in U(2)$.

Imposing the condition $\det g = 1$ is equivalent to

$$\det A(a + \mathrm{e}^{\mathrm{i}\phi} b^\dagger A^{-1} b) \det V = 1. \tag{A.11}$$

From (A.3) and (A.9) we have $\det A = |a|$ and $A^{-1}b = \frac{1}{|a|}b$. Using this, Eq. (A.11) takes the form:

$$|a|\left(a + \mathrm{e}^{+\mathrm{i}\phi}\frac{\|b\|^2}{|a|}\right)\det V = 1.$$

Finally, substituting $a = |a|\,\mathrm{e}^{\mathrm{i}\phi}$ and using (A.3), we obtain:

$$\det V = \mathrm{e}^{-i\phi} = \frac{\bar{a}}{|a|}.$$

We decompose $V = \delta^{-1}S$, where $S \in \mathrm{SU}(2)$ and $\delta^{-2} := \det V$, or $\delta^2 = \frac{a}{|a|}$. Of course, $|\delta| = 1$. Corresponding to the two choices of the square root of $\frac{a}{|a|}$, we choose two open subsets $\mathcal{O}_i \subset \mathcal{O}$,

$$\mathcal{O}_1 := \left\{ \begin{pmatrix} a \\ b_1 \\ b_2 \end{pmatrix} \in \mathcal{O} : \mathrm{phase}(a) \in]-\pi, \pi[\right\},$$

$$\mathcal{O}_2 := \left\{ \begin{pmatrix} a \\ b_1 \\ b_2 \end{pmatrix} \in \mathcal{O} : \mathrm{phase}(a) \in]0, 2\pi[\right\}. \tag{A.12}$$

Then, every element $g \in \pi^{-1}(\mathcal{O}_i) \subset \mathrm{SU}(3)$, can be uniquely represented as

$$g = s_i(\pi(g)) \cdot h_i(g),$$

with s_i being two local cross sections of (A.1) over $\mathcal{O}_i$,

$$S^5 \supset \mathcal{O}_i \ni (a, b) \to s_i(a, b) = \left[\begin{array}{c|c} a & -\delta_i b^\dagger \\ \hline b & \delta_i^{-1}\left(\mathbf{1} - \frac{bb^\dagger}{1+|a|}\right) \end{array}\right] \in \mathrm{SU}(3), \tag{A.13}$$

and

$$h_i(g) = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & S_i(g) \end{array}\right] \subset \mathrm{SU}(3), \qquad S_i(g) \in \mathrm{SU}(2). \tag{A.14}$$

Thus, corresponding to the two choices of the square root, we obtain two local bijective mappings

$$\pi^{-1}(\mathcal{O}_i) \ni g \longrightarrow \chi_i(g) := (\pi(g), (s_i(\pi(g)))^{-1} \cdot g) \in \mathcal{O}_i \times \mathrm{SU}(2).$$

Similarly, we choose the following open neighborhood of $a = 0$:

$$\mathcal{O}_3 := \left\{ \begin{pmatrix} a \\ b_1 \\ b_2 \end{pmatrix} \in S^5 : \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

Then, we find a local cross section s_3 over $\mathcal{O}_3$ such that

$$g = s_3(\pi(g)) \cdot h(g) = \left[\begin{array}{c|c} a & b^\dagger \\ \hline b & -\mathbf{1} + \frac{1-\bar{a}}{\|b\|^2} bb^\dagger \end{array} \right] \cdot \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & S(g) \end{array} \right], \tag{A.15}$$

with $S(g) \in \mathrm{SU}(2)$, and a local bijective mapping

$$\pi^{-1}(\mathcal{O}_3) \ni g \longrightarrow \chi_3(g) := \left(\pi(g), (s_3(\pi(g)))^{-1} \cdot g\right) \in \mathcal{O}_3 \times \mathrm{SU}(2).$$

Proposition A.1. *The local mappings χ_i, $i = 1, 2, 3$, form an atlas of local trivializations of the* SU(2)*-principal bundle* (A.1).

Proof. The proof consists of checking the following obvious statements:

1. The open neighborhoods $\mathcal{O}_i$ cover S^5,

   $$\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 = S^5.$$

2. The mappings

   $$\pi^{-1}(\mathcal{O}_i) \ni g \longrightarrow \chi_i(g) := \left(\pi(g), (s_i(\pi(g)))^{-1} \cdot g\right) \in \mathcal{O}_i \times \mathrm{SU}(2)$$

   are local diffeomorphisms, for $i = 1, 2, 3$.
3. The mappings $\{\chi_i\}$ are compatible with the bundle structure and with the right group action:

   $$pr_1^i \circ \chi_i = \pi, \tag{A.16}$$

   $$(pr_2^i \circ \chi_i)(g \cdot g') = (pr_2^i \circ \chi_i(g)) \cdot g', \tag{A.17}$$

   for $i = 1, 2, 3$, with pr_α^i, $\alpha = 1, 2$, denoting the projection on the first, respectively second factor of $\mathcal{O}_i \times \mathrm{SU}(2)$. □

Appendix B. The relation for T_5^2

The relation for the square of the invariant T_5, referred to in (4.) of Lemma 4.3, is

$$
\begin{aligned}
&(\mathrm{tr}(h^2g^2hg)-\mathrm{tr}(h^2ghg^2))^2\\
&\quad=-27+\mathrm{tr}(h)^2\overline{\mathrm{tr}(h)}^2+18\mathrm{tr}(hg)\overline{\mathrm{tr}(hg)}+\mathrm{tr}(hg)^2\overline{\mathrm{tr}(hg)}^2+\mathrm{tr}(hg^2)^2\overline{\mathrm{tr}(hg^2)}^2\\
&\qquad+18\overline{\mathrm{tr}(hg^2)}\mathrm{tr}(hg^2)-4\mathrm{tr}(h)^3-4\overline{\mathrm{tr}(h)}^3-4\mathrm{tr}(hg^2)^3-4\overline{\mathrm{tr}(g)}^3\\
&\qquad-4\overline{\mathrm{tr}(hg^2)}\mathrm{tr}(h)^2\overline{\mathrm{tr}(hg)}-4\mathrm{tr}(hg^2)\mathrm{tr}(hg)\overline{\mathrm{tr}(h)}^2-4\mathrm{tr}(hg)^2\mathrm{tr}(hg^2)\overline{\mathrm{tr}(g)}\\
&\qquad-4\overline{\mathrm{tr}(hg)}^2\overline{\mathrm{tr}(hg^2)}\mathrm{tr}(g)-4\mathrm{tr}(hg^2)\overline{\mathrm{tr}(hg)}\mathrm{tr}(g)^2-6\mathrm{tr}(hg^2)\overline{\mathrm{tr}(hg)\mathrm{tr}(g)}\\
&\qquad-4\overline{\mathrm{tr}(hg)}^2\overline{\mathrm{tr}(hg^2)}\,\overline{\mathrm{tr}(g)}^2+8\overline{\mathrm{tr}(hg^2)}^2\overline{\mathrm{tr}(hg)\mathrm{tr}(g)}+\mathrm{tr}(hg^2)^2\overline{\mathrm{tr}(hg)}^2\overline{\mathrm{tr}(g)}^2\\
&\qquad+\overline{\mathrm{tr}(hg^2)}^2\mathrm{tr}(hg)^2\mathrm{tr}(g)^2+8\mathrm{tr}(hg^2)^2\mathrm{tr}(hg)\mathrm{tr}(g)-4\mathrm{tr}(hg)\overline{\mathrm{tr}(hg^2)}\,\overline{\mathrm{tr}(g)}^2\\
&\qquad-4\mathrm{tr}(hg)^2\mathrm{tr}(hg^2)\mathrm{tr}(g)^2-4\mathrm{tr}(hg)^2\overline{\mathrm{tr}(h)}\,\overline{\mathrm{tr}(hg^2)}+12\overline{\mathrm{tr}(h)}\,\overline{\mathrm{tr}(hg^2)}\,\overline{\mathrm{tr}(hg)}\\
&\qquad+12\mathrm{tr}(hg^2)\mathrm{tr}(hg)\mathrm{tr}(h)-4\overline{\mathrm{tr}(hg^2)}^2\mathrm{tr}(hg)\mathrm{tr}(h)-4\mathrm{tr}(hg^2)^2\overline{\mathrm{tr}(h)\mathrm{tr}(hg)}\\
&\qquad-4\overline{\mathrm{tr}(hg^2)}^3-2\mathrm{tr}(hg^2)\mathrm{tr}(hg)\overline{\mathrm{tr}(hg)}\,\overline{\mathrm{tr}(hg^2)}-2\mathrm{tr}(hg^2)\overline{\mathrm{tr}(hg^2)}\mathrm{tr}(h)\overline{\mathrm{tr}(h)}\\
&\qquad-4\overline{\mathrm{tr}(hg)}^3-2\mathrm{tr}(hg^2)\mathrm{tr}(h)\overline{\mathrm{tr}(h)\mathrm{tr}(hg)\mathrm{tr}(g)}-4\mathrm{tr}(hg)^3\\
&\qquad-2\overline{\mathrm{tr}(h)\mathrm{tr}(hg)}\mathrm{tr}(h)\mathrm{tr}(hg)+18\mathrm{tr}(h)\overline{\mathrm{tr}(h)}-2\mathrm{tr}(h)\mathrm{tr}(hg)\overline{\mathrm{tr}(hg)}^2\overline{\mathrm{tr}(g)}\mathrm{tr}(g)^2\\
&\qquad-4\overline{\mathrm{tr}(hg)}\mathrm{tr}(h)\overline{\mathrm{tr}(g)}^2-6\overline{\mathrm{tr}(hg)}\mathrm{tr}(h)\mathrm{tr}(g)+12\overline{\mathrm{tr}(hg^2)}\mathrm{tr}(h)\overline{\mathrm{tr}(g)}\\
&\qquad-4\mathrm{tr}(hg^2)\mathrm{tr}(h)^2\mathrm{tr}(g)-4\mathrm{tr}(hg^2)^2\mathrm{tr}(h)\overline{\mathrm{tr}(g)}-4\overline{\mathrm{tr}(hg^2)}^2\overline{\mathrm{tr}(h)}\mathrm{tr}(g)\\
&\qquad+\overline{\mathrm{tr}(g)}^2\mathrm{tr}(g)^2-2\mathrm{tr}(hg^2)\overline{\mathrm{tr}(hg^2)}\mathrm{tr}(h)\overline{\mathrm{tr}(hg)}\mathrm{tr}(g)\\
&\qquad-2\mathrm{tr}(hg)\mathrm{tr}(hg^2)\overline{\mathrm{tr}(hg)}^2\overline{\mathrm{tr}(g)}^2\mathrm{tr}(g)+4\mathrm{tr}(hg^2)\mathrm{tr}(hg)\overline{\mathrm{tr}(hg)}\,\overline{\mathrm{tr}(hg^2)}\,\overline{\mathrm{tr}(g)}\mathrm{tr}(g)\\
&\qquad-2\mathrm{tr}(hg)^2\overline{\mathrm{tr}(hg^2)}\,\overline{\mathrm{tr}(hg)\mathrm{tr}(g)}\mathrm{tr}(g)^2+2\mathrm{tr}(hg)\mathrm{tr}(hg^2)\overline{\mathrm{tr}(h)\mathrm{tr}(hg)\mathrm{tr}(g)}^2\\
&\qquad+4\mathrm{tr}(hg)\mathrm{tr}(hg^2)\overline{\mathrm{tr}(h)\mathrm{tr}(hg)}\mathrm{tr}(g)+2\mathrm{tr}(h)\mathrm{tr}(hg)\overline{\mathrm{tr}(hg^2)}\,\overline{\mathrm{tr}(hg)}\mathrm{tr}(g)^2\\
&\qquad+4\mathrm{tr}(h)\mathrm{tr}(hg)\overline{\mathrm{tr}(hg^2)}\,\overline{\mathrm{tr}(hg)\mathrm{tr}(g)}+4\mathrm{tr}(hg^2)\mathrm{tr}(hg)\mathrm{tr}(h)\overline{\mathrm{tr}(g)}\mathrm{tr}(g)\\
&\qquad+4\overline{\mathrm{tr}(h)}\,\overline{\mathrm{tr}(hg^2)}\,\overline{\mathrm{tr}(hg)\mathrm{tr}(g)}\mathrm{tr}(g)+2\mathrm{tr}(h)\mathrm{tr}(hg^2)\overline{\mathrm{tr}(hg)}^2\overline{\mathrm{tr}(g)}\mathrm{tr}(g)\\
&\qquad+2\mathrm{tr}(hg)^2\overline{\mathrm{tr}(h)}\,\overline{\mathrm{tr}(hg^2)}\,\overline{\mathrm{tr}(g)}\mathrm{tr}(g)+4\mathrm{tr}(hg)^3\overline{\mathrm{tr}(g)}\mathrm{tr}(g)+4\mathrm{tr}(hg)\overline{\mathrm{tr}(hg)\mathrm{tr}(g)}^3\\
&\qquad+4\overline{\mathrm{tr}(hg)}^3\overline{\mathrm{tr}(g)}\mathrm{tr}(g)-2\overline{\mathrm{tr}(hg^2)}\mathrm{tr}(hg)\mathrm{tr}(h)\overline{\mathrm{tr}(h)}\mathrm{tr}(g)-6\mathrm{tr}(hg)\overline{\mathrm{tr}(hg^2)}\mathrm{tr}(g)\\
&\qquad-4\overline{\mathrm{tr}(hg^2)}\,\overline{\mathrm{tr}(h)}^2\overline{\mathrm{tr}(g)}-4\mathrm{tr}(h)\mathrm{tr}(hg)^2\overline{\mathrm{tr}(g)}^2-4\overline{\mathrm{tr}(h)\mathrm{tr}(hg)}^2\overline{\mathrm{tr}(g)}\\
&\qquad-4\mathrm{tr}(hg)\overline{\mathrm{tr}(h)}\mathrm{tr}(g)^2-6\mathrm{tr}(hg)\overline{\mathrm{tr}(h)\mathrm{tr}(g)}-4\mathrm{tr}(h)\mathrm{tr}(hg)^2\mathrm{tr}(g)\\
&\qquad+8\overline{\mathrm{tr}(h)}^2\overline{\mathrm{tr}(hg)}\mathrm{tr}(g)+8\mathrm{tr}(hg)\mathrm{tr}(h)^2\overline{\mathrm{tr}(g)}+\mathrm{tr}(h)^2\overline{\mathrm{tr}(hg)}^2\mathrm{tr}(g)^2\\
&\qquad+\mathrm{tr}(hg)^2\overline{\mathrm{tr}(h)}^2\overline{\mathrm{tr}(g)}^2-4\mathrm{tr}(hg^2)\overline{\mathrm{tr}(h)\mathrm{tr}(g)}^2+12\mathrm{tr}(hg^2)\overline{\mathrm{tr}(h)}\mathrm{tr}(g)\\
&\qquad-4\overline{\mathrm{tr}(hg^2)}\mathrm{tr}(h)\mathrm{tr}(g)^2-4\overline{\mathrm{tr}(h)\mathrm{tr}(hg)}^2\mathrm{tr}(g)^2+4\mathrm{tr}(hg)\overline{\mathrm{tr}(hg)}\mathrm{tr}(g)^3
\end{aligned}
$$

$$
\begin{aligned}
&-2\mathrm{tr}(hg^2)\overline{\mathrm{tr}(hg^2)}\mathrm{tr}(hg)\overline{\mathrm{tr}(h)\mathrm{tr}(g)}-2\mathrm{tr}(hg)^2\overline{\mathrm{tr}(h)\mathrm{tr}(hg)\mathrm{tr}(g)}^2\mathrm{tr}(g)\\
&+4\overline{\mathrm{tr}(h)\mathrm{tr}(hg)}\mathrm{tr}(h)\mathrm{tr}(hg)\overline{\mathrm{tr}(g)}\mathrm{tr}(g)-2\mathrm{tr}(h)\overline{\mathrm{tr}(h)\mathrm{tr}(g)}\mathrm{tr}(g)-4\mathrm{tr}(g)^3\\
&-2\overline{\mathrm{tr}(hg^2)}\mathrm{tr}(hg^2)\overline{\mathrm{tr}(g)}\mathrm{tr}(g)-2\mathrm{tr}(hg^2)\overline{\mathrm{tr}(hg^2)}^2\mathrm{tr}(hg)\mathrm{tr}(g)\\
&-2\mathrm{tr}(hg^2)^2\overline{\mathrm{tr}(hg^2)}\,\overline{\mathrm{tr}(hg)\mathrm{tr}(g)}+2\mathrm{tr}(hg^2)\overline{\mathrm{tr}(hg)\mathrm{tr}(g)}^2\mathrm{tr}(g)\\
&+2\mathrm{tr}(hg)\overline{\mathrm{tr}(hg^2)}\,\overline{\mathrm{tr}(g)}\mathrm{tr}(g)^2+2\mathrm{tr}(hg)\mathrm{tr}(hg^2)\overline{\mathrm{tr}(hg)}^2\,\overline{\mathrm{tr}(g)}\\
&+2\mathrm{tr}(hg)^2\overline{\mathrm{tr}(hg^2)}\,\overline{\mathrm{tr}(hg)}\mathrm{tr}(g)+\mathrm{tr}(hg)^2\overline{\mathrm{tr}(hg)}^2\,\overline{\mathrm{tr}(g)}^2\mathrm{tr}(g)^2\\
&-2\mathrm{tr}(hg)^2\overline{\mathrm{tr}(hg)}^2\,\overline{\mathrm{tr}(g)}\mathrm{tr}(g)-2\mathrm{tr}(hg)\overline{\mathrm{tr}(hg)\mathrm{tr}(g)}^2\mathrm{tr}(g)^2\\
&-8\mathrm{tr}(hg)\overline{\mathrm{tr}(hg)\mathrm{tr}(g)}\mathrm{tr}(g)+2\mathrm{tr}(h)\mathrm{tr}(hg)\overline{\mathrm{tr}(hg)}^2\mathrm{tr}(g)+2\mathrm{tr}(hg)\overline{\mathrm{tr}(h)\mathrm{tr}(g)}^2\mathrm{tr}(g)\\
&+2\overline{\mathrm{tr}(hg)}\mathrm{tr}(h)\overline{\mathrm{tr}(g)}\mathrm{tr}(g)^2+2\mathrm{tr}(hg)^2\overline{\mathrm{tr}(h)\mathrm{tr}(hg)\mathrm{tr}(g)}-2\mathrm{tr}(h)^2\overline{\mathrm{tr}(h)\mathrm{tr}(hg)}\mathrm{tr}(g)\\
&-2\mathrm{tr}(hg)\mathrm{tr}(h)\overline{\mathrm{tr}(h)}^2\,\overline{\mathrm{tr}(g)}+18\overline{\mathrm{tr}(g)}\mathrm{tr}(g)-4\mathrm{tr}(h)\mathrm{tr}(hg^2)\overline{\mathrm{tr}(hg)}^2.
\end{aligned}
$$

It can be derived in the following way. Consider the invariant functions $\mathrm{tr}(hghgghghhggh)$ and $\mathrm{tr}(hghgghhgghgh)$ of order 12. The sum of them can be expressed in terms of generators $T_1, \ldots, T_5$ in two different ways. First, we use the trace identity (2.6) for $k = 4$ and $g_1 = gh$, $g_2 = gg$, $g_3 = hg$, $g_4 = hhgghh$ to express $\mathrm{tr}(hghgghghhggh)$ in terms of traces of lower order. Next, we use the trace identity (2.6) for $k = 4$ and $g_1 = hh$, $g_2 = gh$, $g_3 = gghhgg$, $g_4 = hg$ to express $\mathrm{tr}(hghgghhgghgh)$. It turns out that in both cases (which are actually equivalent, because one is obtained from the other by interchanging g with h), we obtain expressions which can be simplified using standard techniques from Section 4. The final expressions in terms of generators do not depend on T_5.

On the other hand we observe that the sum

$$
\begin{aligned}
&\mathrm{tr}(hghgghghhggh)+\mathrm{tr}(hghgghhgghgh)\\
&\quad=\mathrm{tr}((hg)^2(gh)^2(hg)(gh))+\mathrm{tr}((hg)^2(gh)(hg)(gh)^2)
\end{aligned}
$$

can be expressed in terms of invariants of lower order using formula (4.9) (we substitute $h \to hg$, $g \to gh$). In this case, we obtain a different formula containing T_5^2. Taking the difference of these two expressions yields the above relation.

All calculations described above were made by a computer program written under *Maple 8.00*. It is worth mentioning that this program automatically generates polynomial expression in terms of generators for any trace function (at least up to order 12) using only standard techniques, namely fundamental trace identities and appropriate substitutions in the Cayley equation.

Finally, let us mention that, once the relation has been found, it can be checked by direct calculation.

Appendix C. The polynomials I_0, I_1^R and I_2^R

$$
\begin{aligned}
&I_0(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)\\
&= -x_1^4x_5^4 - 2x_1^4x_5^2x_6^2 - x_1^4x_6^4 - 2x_1^2x_2^2x_5^4 - 4x_1^2x_2^2x_5^2x_6^2 - 2x_1^2x_2^2x_6^4 - x_2^4x_5^4\\
&\quad - 2x_2^4x_5^2x_6^2 - x_2^4x_6^4 + 4x_1^3x_3x_5^3 + 4x_1^3x_3x_5x_6^2 + 4x_1^3x_4x_5^2x_6 + 4x_1^3x_4x_6^3\\
&\quad + 4x_1^3x_5^3x_7 + 4x_1^3x_5^2x_6x_8 + 4x_1^3x_5x_6^2x_7 + 4x_1^3x_6^3x_8 + 4x_1^2x_2x_3x_5^2x_6\\
&\quad + 4x_1^2x_2x_3x_6^3 - 4x_1^2x_2x_4x_5^3 - 4x_1^2x_2x_4x_5x_6^2 + 4x_1^2x_2x_5^3x_8 - 4x_1^2x_2x_5^2x_6x_7\\
&\quad + 4x_1^2x_2x_5x_6^2x_8 - 4x_1^2x_2x_6^3x_7 + 4x_1x_2^2x_3x_5^3 + 4x_1x_2^2x_3x_5x_6^2 + 4x_1x_2^2x_4x_5^2x_6\\
&\quad + 4x_1x_2^2x_4x_6^3 + 4x_1x_2^2x_5^3x_7 + 4x_1x_2^2x_5^2x_6x_8 + 4x_1x_2^2x_5x_6^2x_7 + 4x_1x_2^2x_6^3x_8\\
&\quad + 4x_2^3x_3x_5^2x_6 + 4x_2^3x_3x_6^3 - 4x_2^3x_4x_5^3 - 4x_2^3x_4x_5x_6^2 + 4x_2^3x_5^3x_8 - 4x_2^3x_5^2x_6x_7\\
&\quad + 4x_2^3x_5x_6^2x_8 - 4x_2^3x_6^3x_7 + 2x_1^4x_5^2 + 2x_1^4x_6^2 + 4x_1^2x_2^2x_5^2 + 4x_1^2x_2^2x_6^2 - 6x_1^2x_3^2x_5^2\\
&\quad - 2x_1^2x_3^2x_6^2 - 8x_1^2x_3x_4x_5x_6 - 8x_1^2x_3x_5^2x_7 - 8x_1^2x_3x_5x_6x_8 - 2x_1^2x_4^2x_5^2\\
&\quad - 6x_1^2x_4^2x_6^2 - 8x_1^2x_4x_5x_6x_7 - 8x_1^2x_4x_6^2x_8 + 2x_1^2x_5^4 + 4x_1^2x_5^2x_6^2 - 6x_1^2x_5^2x_7^2\\
&\quad - 2x_1^2x_5^2x_8^2 - 8x_1^2x_5x_6x_7x_8 + 2x_1^2x_6^4 - 2x_1^2x_6^2x_7^2 - 6x_1^2x_6^2x_8^2 - 8x_1x_2x_3^2x_5x_6\\
&\quad + 8x_1x_2x_3x_4x_5^2 - 8x_1x_2x_3x_4x_6^2 - 8x_1x_2x_3x_5^2x_8 - 8x_1x_2x_3x_6^2x_8\\
&\quad + 8x_1x_2x_4^2x_5x_6 + 8x_1x_2x_4x_5^2x_7 + 8x_1x_2x_4x_6^2x_7 - 8x_1x_2x_5^2x_7x_8\\
&\quad + 8x_1x_2x_5x_6x_7^2 - 8x_1x_2x_5x_6x_8^2 + 8x_1x_2x_6^2x_7x_8 + 2x_2^4x_5^2 + 2x_2^4x_6^2 - 2x_2^2x_3^2x_5^2\\
&\quad - 6x_2^2x_3^2x_6^2 + 8x_2^2x_3x_4x_5x_6 - 8x_2^2x_3x_5x_6x_8 + 8x_2^2x_3x_6^2x_7 - 6x_2^2x_4^2x_5^2\\
&\quad - 2x_2^2x_4^2x_6^2 + 8x_2^2x_4x_5^2x_8 - 8x_2^2x_4x_5x_6x_7 + 2x_2^2x_5^4 + 4x_2^2x_5^2x_6^2 - 2x_2^2x_5^2x_7^2\\
&\quad - 6x_2^2x_5^2x_8^2 + 8x_2^2x_5x_6x_7x_8 + 2x_2^2x_6^4 - 6x_2^2x_6^2x_7^2 - 2x_2^2x_6^2x_8^2 - 4x_1^3x_3x_5\\
&\quad - 4x_1^3x_4x_6 - 8x_1^3x_5^2 - 4x_1^3x_5x_7 - 8x_1^3x_6^2 - 4x_1^3x_6x_8 - 4x_1^2x_2x_3x_6\\
&\quad + 4x_1^2x_2x_4x_5 - 4x_1^2x_2x_5x_8 + 4x_1^2x_2x_6x_7 + 8x_1^2x_3x_5^2 - 8x_1^2x_3x_5x_7 - 8x_1^2x_3x_6^2\\
&\quad + 8x_1^2x_3x_6x_8 - 16x_1^2x_4x_5x_6 + 8x_1^2x_4x_5x_8 + 8x_1^2x_4x_6x_7 - 8x_1^2x_5^3 + 8x_1^2x_5^2x_7\\
&\quad + 24x_1^2x_5x_6^2 - 16x_1^2x_5x_6x_8 - 8x_1^2x_6^2x_7 - 4x_1x_2^2x_3x_5 - 4x_1x_2^2x_4x_6 + 24x_1x_2^2x_5^2\\
&\quad - 4x_1x_2^2x_5x_7 + 24x_1x_2^2x_6^2 - 4x_1x_2^2x_6x_8 + 32x_1x_2x_3x_5x_6 + 16x_1x_2x_4x_5^2\\
&\quad - 16x_1x_2x_4x_6^2 - 16x_1x_2x_5^2x_8 - 32x_1x_2x_5x_6x_7 + 16x_1x_2x_6^2x_8 + 4x_1x_3^3x_5\\
&\quad + 4x_1x_3^2x_4x_6 + 4x_1x_3^2x_5x_7 + 4x_1x_3^2x_6x_8 + 4x_1x_3x_4^2x_5 - 4x_1x_3x_5^3\\
&\quad - 8x_1x_3x_5^2x_7 - 4x_1x_3x_5x_6^2 + 4x_1x_3x_5x_7^2 + 4x_1x_3x_5x_8^2 - 8x_1x_3x_6^2x_7\\
&\quad + 4x_1x_4^3x_6 + 4x_1x_4^2x_5x_7 + 4x_1x_4^2x_6x_8 - 4x_1x_4x_5^2x_6 - 8x_1x_4x_5^2x_8 - 4x_1x_4x_6^3
\end{aligned}
$$

$$
\begin{aligned}
&- 8x_1x_4x_6^2x_8 + 4x_1x_4x_6x_7^2 + 4x_1x_4x_6x_8^2 - 4x_1x_5^3x_7 - 4x_1x_5^2x_6x_8 \\
&- 4x_1x_5x_6^2x_7 + 4x_1x_5x_7^3 + 4x_1x_5x_7x_8^2 - 4x_1x_6^3x_8 + 4x_1x_6x_7^2x_8 + 4x_1x_6x_8^3 \\
&- 4x_2^3x_3x_6 + 4x_2^3x_4x_5 - 4x_2^3x_5x_8 + 4x_2^3x_6x_7 - 8x_2^2x_3x_5^2 - 8x_2^2x_3x_5x_7 \\
&+ 8x_2^2x_3x_6^2 + 8x_2^2x_3x_6x_8 + 16x_2^2x_4x_5x_6 + 8x_2^2x_4x_5x_8 + 8x_2^2x_4x_6x_7 - 8x_2^2x_5^3 \\
&- 8x_2^2x_5^2x_7 + 24x_2^2x_5x_6^2 + 16x_2^2x_5x_6x_8 + 8x_2^2x_6^2x_7 + 4x_2x_3^3x_6 - 4x_2x_3^2x_4x_5 \\
&+ 4x_2x_3^2x_5x_8 - 4x_2x_3^2x_6x_7 + 4x_2x_3x_4^2x_6 - 4x_2x_3x_5^2x_6 + 8x_2x_3x_5^2x_8 \\
&- 4x_2x_3x_6^3 + 8x_2x_3x_6^2x_8 + 4x_2x_3x_6x_7^2 + 4x_2x_3x_6x_8^2 - 4x_2x_4^3x_5 + 4x_2x_4^2x_5x_8 \\
&- 4x_2x_4^2x_6x_7 + 4x_2x_4x_5^3 - 8x_2x_4x_5^2x_7 + 4x_2x_4x_5x_6^2 - 4x_2x_4x_5x_7^2 \\
&- 4x_2x_4x_5x_8^2 - 8x_2x_4x_6^2x_7 - 4x_2x_5^3x_8 + 4x_2x_5^2x_6x_7 - 4x_2x_5x_6^2x_8 \\
&+ 4x_2x_5x_7^2x_8 + 4x_2x_5x_8^3 + 4x_2x_6^3x_7 - 4x_2x_6x_7^3 - 4x_2x_6x_7x_8^2 - x_1^4 - 2x_1^2x_2^2 \\
&+ 2x_1^2x_3^2 + 8x_1^2x_3x_5 + 8x_1^2x_3x_7 + 2x_1^2x_4^2 + 8x_1^2x_4x_6 + 8x_1^2x_4x_8 + 8x_1^2x_5^2 \\
&+ 8x_1^2x_5x_7 + 8x_1^2x_6^2 + 8x_1^2x_6x_8 + 2x_1^2x_7^2 + 2x_1^2x_8^2 - 16x_1x_2x_3x_6 \\
&+ 16x_1x_2x_3x_8 + 16x_1x_2x_4x_5 - 16x_1x_2x_4x_7 - 16x_1x_2x_5x_8 + 16x_1x_2x_6x_7 \\
&- 16x_1x_3^2x_5 + 8x_1x_3^2x_7 + 32x_1x_3x_4x_6 - 16x_1x_3x_4x_8 + 8x_1x_3x_5^2 - 8x_1x_3x_6^2 \\
&+ 8x_1x_3x_7^2 - 8x_1x_3x_8^2 + 16x_1x_4^2x_5 - 8x_1x_4^2x_7 - 16x_1x_4x_5x_6 - 16x_1x_4x_7x_8 \\
&+ 8x_1x_5^2x_7 - 16x_1x_5x_6x_8 - 16x_1x_5x_7^2 + 16x_1x_5x_8^2 - 8x_1x_6^2x_7 + 32x_1x_6x_7x_8 \\
&- x_2^4 + 2x_2^2x_3^2 - 8x_2^2x_3x_5 - 8x_2^2x_3x_7 + 2x_2^2x_4^2 - 8x_2^2x_4x_6 - 8x_2^2x_4x_8 + 8x_2^2x_5^2 \\
&- 8x_2^2x_5x_7 + 8x_2^2x_6^2 - 8x_2^2x_6x_8 + 2x_2^2x_7^2 + 2x_2^2x_8^2 - 16x_2x_3^2x_6 - 8x_2x_3^2x_8 \\
&- 32x_2x_3x_4x_5 - 16x_2x_3x_4x_7 - 16x_2x_3x_5x_6 + 16x_2x_3x_7x_8 + 16x_2x_4^2x_6 \\
&+ 8x_2x_4^2x_8 - 8x_2x_4x_5^2 + 8x_2x_4x_6^2 + 8x_2x_4x_7^2 - 8x_2x_4x_8^2 + 8x_2x_5^2x_8 \\
&+ 16x_2x_5x_6x_7 + 32x_2x_5x_7x_8 - 8x_2x_6^2x_8 + 16x_2x_6x_7^2 - 16x_2x_6x_8^2 - x_3^4 \\
&- 2x_3^2x_4^2 + 2x_3^2x_5^2 + 8x_3^2x_5x_7 + 2x_3^2x_6^2 - 8x_3^2x_6x_8 + 2x_3^2x_7^2 + 2x_3^2x_8^2 \\
&+ 16x_3x_4x_5x_8 + 16x_3x_4x_6x_7 + 8x_3x_5^2x_7 + 16x_3x_5x_6x_8 + 8x_3x_5x_7^2 - 8x_3x_5x_8^2 \\
&- 8x_3x_6^2x_7 + 16x_3x_6x_7x_8 - x_4^4 + 2x_4^2x_5^2 - 8x_4^2x_5x_7 + 2x_4^2x_6^2 + 8x_4^2x_6x_8 \\
&+ 2x_4^2x_7^2 + 2x_4^2x_8^2 - 8x_4x_5^2x_8 + 16x_4x_5x_6x_7 + 16x_4x_5x_7x_8 + 8x_4x_6^2x_8 \\
&- 8x_4x_6x_7^2 + 8x_4x_6x_8^2 - x_5^4 - 2x_5^2x_6^2 + 2x_5^2x_7^2 + 2x_5^2x_8^2 - x_6^4 + 2x_6^2x_7^2 \\
&+ 2x_6^2x_8^2 - x_7^4 - 2x_7^2x_8^2 - x_8^4 + 8x_1^3 - 24x_1x_2^2 + 12x_1x_3x_5 - 24x_1x_3x_7 \\
&+ 12x_1x_4x_6 - 24x_1x_4x_8 + 12x_1x_5x_7 + 12x_1x_6x_8 + 12x_2x_3x_6 + 24x_2x_3x_8 \\
&-12x_2x_4x_5 - 24x_2x_4x_7 + 12x_2x_5x_8 - 12x_2x_6x_7 + 8x_3^3 - 24x_3x_4^2
\end{aligned}
$$

$$
\begin{aligned}
&- 24x_3x_5x_7 + 24x_3x_6x_8 + 24x_4x_5x_8 + 24x_4x_6x_7 + 8x_5^3 - 24x_5x_6^2 + 8x_7^3\\
&-24x_7x_8^2 - 18x_1^2 - 18x_2^2 - 18x_3^2 - 18x_4^2 - 18x_5^2 - 18x_6^2 - 18x_7^2 - 18x_8^2 + 27,
\end{aligned}
$$

$$
\begin{aligned}
&I_1^R(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)\\
&= 27 - 9x_8^2 - 9x_7^2 - 9x_3^2 - 9x_4^2 - 6x_2x_6x_7 + 2x_7^3 - 6x_7x_8^2 - 9x_6^2 - 9x_5^2\\
&\quad + 6x_1x_5x_7 + 6x_2x_5x_8 - 8x_1x_2x_3x_6 + 8x_1x_2x_4x_5 + 2x_5^3 - 6x_5x_6^2 + 4x_1^2x_3x_5\\
&\quad - 4x_2^2x_3x_5 + 6x_1x_3x_5 + 6x_2x_3x_6 - 6x_2x_4x_5 + 4x_1^2x_4x_6 - 4x_2^2x_4x_6 + 2x_3^3\\
&\quad - 6x_3x_4^2 + 4x_1x_2x_4x_5^2 - 4x_2x_3x_5x_6 - 4x_1^2x_4x_5x_6 + 4x_2^2x_4x_5x_6 - 4x_1x_4x_5x_6\\
&\quad - 4x_1x_2x_4x_6^2 - 4x_1^2x_5x_6x_8 + 4x_2^2x_5x_6x_8 + 4x_1x_4^2x_5 + 2x_1^2x_3x_6x_8\\
&\quad + 2x_2^2x_3x_6x_8 + 2x_1^2x_4x_6x_7 + 2x_2^2x_4x_6x_7 + 2x_1^2x_4x_5x_8 + 2x_2^2x_4x_5x_8\\
&\quad + 4x_1^2x_5x_7 + 6x_1x_4x_6 - 4x_2^2x_5x_7 + 4x_1^2x_6x_8 - 4x_2^2x_6x_8 - 4x_1x_3^2x_5\\
&\quad - 4x_2x_3^2x_6 + 4x_2x_4^2x_6 + 8x_1x_2x_6x_7 - 8x_1x_2x_5x_8 - 2x_1^2x_3x_5x_7 - 2x_2^2x_3x_5x_7\\
&\quad + x_1^4x_6^2 + x_2^4x_6^2 - 4x_1^3x_6^2 + 2x_1^2x_2^2x_5^2 + x_2^4x_5^2 - 4x_1^3x_5^2 + 2x_1x_3^2x_7 - 2x_1x_4^2x_7\\
&\quad - 2x_2x_3^2x_8 + 2x_2x_4^2x_8 + 2x_1x_3x_7^2 - 2x_1x_3x_8^2 + 2x_2x_4x_7^2 - 2x_2x_4x_8^2 + x_1^2x_7^2\\
&\quad + x_2^2x_7^2 + x_1^2x_8^2 + x_2^2x_8^2 - 8x_1x_2x_4x_7 + 8x_1x_2x_3x_8 - 4x_2x_3x_4x_7\\
&\quad - 4x_1x_3x_4x_8 - 4x_1x_4x_7x_8 + 4x_2x_3x_7x_8 - 8x_1x_2x_5x_6x_7 + 4x_1^2x_3x_7\\
&\quad - 4x_2^2x_3x_7 - 12x_1x_3x_7 - 12x_2x_4x_7 + 12x_2x_3x_8 + 4x_1^2x_4x_8 - 4x_2^2x_4x_8\\
&\quad - 12x_1x_4x_8 + 12x_1x_2^2x_5^2 + 2x_1^2x_2^2x_6^2 + 12x_1x_2^2x_6^2 - 2x_1^3x_5x_7 - 2x_1x_2^2x_5x_7\\
&\quad - 2x_1^2x_2x_5x_8 + 2x_1^2x_2x_6x_7 - 2x_1x_2^2x_6x_8 + 8x_1x_6x_7x_8 + 8x_2x_5x_7x_8\\
&\quad + 4x_2x_5x_6x_7 - 4x_1x_2x_5^2x_8 + 4x_1x_2x_6^2x_8 - 4x_1x_5x_6x_8 - 4x_1x_5x_7^2 + 4x_1x_5x_8^2\\
&\quad + 4x_2x_6x_7^2 - 4x_2x_6x_8^2 - 2x_2^3x_5x_8 + 2x_2^3x_6x_7 - 2x_1^3x_6x_8 + 2x_1x_5^2x_7 - 2x_1x_6^2x_7\\
&\quad + 2x_2x_5^2x_8 - 2x_2x_6^2x_8 + 4x_1^2x_5^2 + 4x_2^2x_5^2 + 4x_1^2x_6^2 + 4x_2^2x_6^2 + 2x_2x_4x_6^2\\
&\quad + 2x_1^2x_5^2x_7 - 2x_2^2x_5^2x_7 - 2x_1^2x_6^2x_7 + 2x_2^2x_6^2x_7 + 8x_1x_2x_3x_5x_6 + 2x_1^2x_3x_5^2\\
&\quad - 2x_2^2x_3x_5^2 + 2x_1x_3x_5^2 - 2x_2x_4x_5^2 - 2x_1^2x_3x_6^2 + 2x_2^2x_3x_6^2 - 2x_1x_3x_6^2\\
&\quad + 8x_1x_3x_4x_6 - 8x_2x_3x_4x_5 + 6x_3x_6x_8 + 6x_4x_5x_8 - 6x_3x_5x_7 + 6x_4x_6x_7\\
&\quad - 2x_1^3x_4x_6 - 2x_1x_2^2x_3x_5 - 2x_1^2x_2x_3x_6 + 2x_1^2x_2x_4x_5 - 2x_1x_2^2x_4x_6 + x_1^2x_3^2\\
&\quad + x_2^2x_3^2 - 2x_1^3x_3x_5 - 2x_2^3x_3x_6 + 2x_2^3x_4x_5 + 6x_1^2x_5x_6^2 + 6x_2^2x_5x_6^2 - 2x_1^2x_5^3\\
&\quad - 2x_2^2x_5^3 + x_1^2x_4^2 + x_2^2x_4^2 - 18x_1^2 - 18x_2^2 + 6x_1x_6x_8 - 24x_1x_2^2 + 8x_1^3\\
&\quad + x_1^4x_5^2 - 2x_1^2x_2^2 - x_1^4 - x_2^4,
\end{aligned}
$$

$$
\begin{aligned}
&I_2^R(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)\\
&\quad = 27 - 2x_1x_3^2x_5^2 + 4x_2x_3^2x_5x_6 - 4x_2x_4^2x_5x_6 + 2x_1x_4^2x_5^2 + 2x_1x_3^2x_6^2 - 2x_1x_4^2x_6^2\\
&\qquad + 2x_1^2x_3x_5x_6^2 - 2x_2^2x_3x_5x_6^2 - 4x_1x_2x_3x_6^3 + 2x_1^2x_4x_5^2x_6 - 2x_2^2x_4x_5^2x_6\\
&\qquad + 4x_1x_2x_4x_5^3 + 6x_1x_2^2x_5^3 + 6x_1^3x_5x_6^2 - 6x_2^3x_5^2x_6 - 6x_1^2x_2x_6^3 - 2x_1^3x_5^3\\
&\qquad + 2x_2^3x_6^3 - 18x_1x_2^2x_5x_6^2 + 18x_1^2x_2x_5^2x_6 - 4x_1x_2x_3x_5^2x_6 + 4x_1x_2x_4x_5x_6^2\\
&\qquad + 2x_1^2x_3x_5^3 - 2x_2^2x_3x_5^3 + 2x_1^2x_4x_6^3 - 2x_2^2x_4x_6^3 - 9x_8^2 - 9x_7^2 - 18x_3^2 - 18x_4^2\\
&\qquad - 12x_2x_6x_7 + 2x_7^3 - 6x_7x_8^2 - 9x_6^2 - 9x_5^2 + 12x_1x_5x_7 + 12x_2x_5x_8\\
&\qquad - 16x_1x_2x_3x_6 + 16x_1x_2x_4x_5 + 2x_5^3 - 6x_5x_6^2 - 2x_3^2x_4^2 + 8x_1^2x_3x_5 - 8x_2^2x_3x_5\\
&\qquad + 6x_1x_3x_5 + 6x_2x_3x_6 - 6x_2x_4x_5 + 8x_1^2x_4x_6 - 8x_2^2x_4x_6 + 8x_3^3 - 24x_3x_4^2\\
&\qquad - 16x_2x_3x_5x_6 - 16x_1x_4x_5x_6 - 12x_1^2x_5x_6x_8 + 12x_2^2x_5x_6x_8 + 8x_1x_4^2x_5\\
&\qquad + 4x_1^2x_3x_6x_8 + 4x_2^2x_3x_6x_8 + 4x_1^2x_4x_6x_7 + 4x_2^2x_4x_6x_7 + 4x_1^2x_4x_5x_8\\
&\qquad + 4x_2^2x_4x_5x_8 - 4x_1x_3x_5^2x_7 - 4x_2x_4x_5^2x_7 + 6x_1x_4x_6 - 8x_1x_3^2x_5 - 8x_2x_3^2x_6\\
&\qquad + 8x_2x_4^2x_6 + 4x_2x_3x_5^2x_8 - 4x_1x_4x_5^2x_8 + 4x_2x_3x_6^2x_8 - 4x_1x_4x_6^2x_8\\
&\qquad - 4x_1^2x_3x_5x_7 - 4x_2^2x_3x_5x_7 + 2x_3x_5^2x_7 - 2x_3x_6^2x_7 + 2x_1x_3x_4^2x_5 - 2x_2x_3^2x_4x_5\\
&\qquad + 2x_1x_3^2x_4x_6 + 2x_2x_3x_4^2x_6 - 2x_2x_3x_5^2x_6 - 2x_1x_3x_5x_6^2 - 2x_1x_4x_5^2x_6\\
&\qquad + 2x_2x_4x_5x_6^2 + 4x_4x_5x_6x_7 + 4x_3x_5x_6x_8 + x_3^2x_5^2 + x_4^2x_5^2 + x_3^2x_6^2 + x_4^2x_6^2\\
&\qquad + 4x_1x_3^2x_7 - 4x_1x_4^2x_7 - 4x_2x_3^2x_8 + 4x_2x_4^2x_8 + 2x_1x_3x_7^2 - 2x_1x_3x_8^2\\
&\qquad + 2x_2x_4x_7^2 - 2x_2x_4x_8^2 - 4x_1x_2x_4x_7 + 4x_1x_2x_3x_8 - 8x_2x_3x_4x_7 - 8x_1x_3x_4x_8\\
&\qquad - 4x_1x_4x_7x_8 + 4x_2x_3x_7x_8 - 24x_1x_2x_5x_6x_7 + 2x_1^2x_3x_7 - 2x_2^2x_3x_7\\
&\qquad - 12x_1x_3x_7 - 12x_2x_4x_7 + 12x_2x_3x_8 + 2x_1^2x_4x_8 - 2x_2^2x_4x_8 - 12x_1x_4x_8\\
&\qquad - x_1^2x_3^2x_5^2 - x_2^2x_3^2x_5^2 - x_1^2x_4^2x_5^2 - x_2^2x_4^2x_5^2 - x_1^2x_3^2x_6^2 - x_2^2x_3^2x_6^2 - x_1^2x_4^2x_6^2\\
&\qquad - x_2^2x_4^2x_6^2 + 2x_1x_3^3x_5 - 2x_2x_4^3x_5 + 2x_1x_4^3x_6 + 2x_2x_3^3x_6 + 12x_1x_6x_7x_8\\
&\qquad + 12x_2x_5x_7x_8 - 12x_1x_2x_5^2x_8 + 12x_1x_2x_6^2x_8 + 4x_3^2x_5x_7 - 4x_4^2x_5x_7\\
&\qquad - 6x_1x_5x_7^2 + 6x_1x_5x_8^2 + 6x_2x_6x_7^2 - 6x_2x_6x_8^2 + 4x_3x_6x_7x_8 + 4x_4x_5x_7x_8\\
&\qquad + 8x_3x_4x_5x_8 + 8x_3x_4x_6x_7 - 3x_1^2x_5^2 - 3x_2^2x_5^2 - 3x_1^2x_6^2 - 3x_2^2x_6^2 + 2x_3x_5x_7^2\\
&\qquad + 8x_2x_4x_6^2 + 6x_1^2x_5^2x_7 - 6x_2^2x_5^2x_7 - 6x_1^2x_6^2x_7 + 6x_2^2x_6^2x_7 - 2x_3x_5x_8^2\\
&\qquad - 2x_4x_6x_7^2 + 2x_4x_6x_8^2 - 4x_3^2x_6x_8 + 4x_4^2x_6x_8 + 8x_1x_3x_5^2 - 8x_2x_4x_5^2\\
&\qquad - 8x_1x_3x_6^2 - 4x_1x_3x_6^2x_7 - 4x_2x_4x_6^2x_7 + 16x_1x_3x_4x_6 - 16x_2x_3x_4x_5 - x_3^4
\end{aligned}
$$

$$
\begin{aligned}
&- x_4^4 + x_3^2x_8^2 + x_4^2x_8^2 + 12x_3x_6x_8 + 12x_4x_5x_8 - 2x_4x_5^2x_8 + 2x_4x_6^2x_8\\
&- 12x_3x_5x_7 + 12x_4x_6x_7 - 2x_1^3x_4x_6 - 2x_1x_3x_5^3 - 2x_2x_3x_6^3 + 2x_2x_4x_5^3\\
&- 2x_1x_4x_6^3 - 2x_1x_2^2x_3x_5 - 2x_1^2x_2x_3x_6 + 2x_1^2x_2x_4x_5 - 2x_1x_2^2x_4x_6 + x_3^2x_7^2\\
&+ x_4^2x_7^2 + x_1^2x_3^2 + x_2^2x_3^2 - 2x_1^3x_3x_5 - 2x_2^3x_3x_6 + 2x_2^3x_4x_5 + x_1^2x_4^2 + x_2^2x_4^2\\
&- 9x_1^2 - 9x_2^2 + 12x_1x_6x_8 - 6x_1x_2^2 + 2x_1^3 + 2x_1^3x_3x_5^2 - 2x_1^3x_3x_6^2\\
&- 8x_1x_3x_4x_5x_6 - 4x_1^2x_2x_3x_5x_6 - 4x_1x_2^2x_4x_5x_6 + 2x_1^2x_2x_4x_6^2 - 2x_1x_2^2x_3x_6^2\\
&+ 2x_1x_2^2x_3x_5^2 - 2x_1^2x_2x_4x_5^2 + 8x_1x_2x_3x_4x_5 - 2x_1^2x_3^2x_5 + 2x_2^2x_3^2x_5 + 2x_1^2x_4^2x_5\\
&- 2x_2^2x_4^2x_5 - 2x_2^3x_4x_5^2 + 2x_2^3x_4x_6^2 - 4x_2^3x_3x_5x_6 - 4x_1^3x_4x_5x_6 - 4x_2x_3x_4x_5^2\\
&+ 4x_2x_3x_4x_6^2 + 4x_1x_2x_3^2x_6 - 4x_1x_2x_4^2x_6 + 4x_1^2x_3x_4x_6 - 4x_2^2x_3x_4x_6.
\end{aligned}
$$

The polynomials I_1^R and I_2^R are both inhomogeneous polynomials of total degree 6. I_1^R is a polynomial of degrees 4, 4, 3, 2, 3, 2, 3, 2 and I_2^R is of degrees 3, 3, 4, 4, 3, 3, 3, 2 in variables $x_1, \ldots, x_8$ respectively.

The non-reduced polynomials I_1 and I_2 are both inhomogeneous real polynomial of degree 8. I_1 is of degree 4 in every variable $x_1, \ldots, x_8$ and I_2 is of degree 4 in the variables $x_1, x_2, x_5, x_6, x_7, x_8$ and of degree 3 in the variables x_3, x_4.

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